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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

# APPLICATION OF A DEFINITE INTEGRAL INVOLVING BESSEL'S FUNCTIONS TO THE SELF-INDUCTANCE OF SOLENOIDS.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society, December 29, 1905).

THE construction in the Physical Laboratory of Clark University of two primary standards of self-inductance by Dr. J. G. Coffin and his development of satisfactory formulas for the calculation of their values* led the writer to attempt to find an expression of a different sort for this quantity, namely the self-inductance of a current sheet in the form of a circular cylinder, practically represented by a wire wound in a close solenoid of a single layer.† All the methods that have been previously used have involved elliptic integrals or developments in spherical harmonics. A quite different mode of approaching the problem was suggested by a paper by Nagaoka,‡ in which use is made of the following integral given by H. Weber§ for the value of the potential of a circular disk of radius $a$ and of unit surface density at a point at a distance $z$ from its plane and $r$ from the axis of symmetry:

$$(1) \qquad V = 2\pi a \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_0(\lambda r) J_1(\lambda a) d\lambda \qquad (z > 0).$$

* J. G. Coffin, *Bulletin* of the Bureau of Standards, vol. 2, No. 1 (June, 1906).

† Since this paper was read several papers by E. B. Rosa and L. Cohen on the *mutual* inductance of such solenoids have appeared in the *Bulletin* of the Bureau of Standards.

‡ Nagaoka, "On the potential and lines of force of a circular current," *Phil. Mag.*, vol. 6, p. 19, 1903.

§ H. Weber, "Ueber die Bessel'schen Funktionen und ihre Anwendungen auf die Theorie der elektrischen Ströme." *Crelle's Journal*, vol. 75, p. 75.

The magnetic field due to a current $I$ in a close solenoid of $m$ turns per unit of length parallel to the axis, which we will take as the $z$-axis, is $4\pi mI$, if the solenoid is of infinite length. If not, we can find the effect of the ends by placing at the positive and negative ends disks with surface density $\pm mI$ respectively. We have then for the magnetic potential at points inside the solenoid

$$\Omega = mI(V_1 - V_2 - 4\pi z), \tag{2}$$

and for the field

$$H = -\frac{\partial \Omega}{\partial z} = mI\left(4\pi - \frac{\partial V_1}{\partial z} + \frac{\partial V_2}{\partial z}\right), \tag{3}$$

in which $V_2$ denotes the value obtained by putting in (1) for $r$ the distance $z_2$ of the point from the negative end of the cylinder, and $V_1$ the value obtained by putting the distance from the positive end $z_1 = l - z_2$, where $l$ is the length of the solenoid.

Differentiating under the integral sign, since

$$\frac{\partial}{\partial z_2} = -\frac{\partial}{\partial z_1} = \frac{\partial}{\partial z},$$

we find the flux through a cross-section of the cylinder by multiplying by $2\pi r dr$ and integrating from 0 to $a$,

$$2\pi \int_0^a H r dr = mI\left\{4\pi^2 a^2 - 4\pi^2 a \int_0^a r dr \int_0^\infty e^{-\lambda z_1} J_0(\lambda r) J_1(\lambda a) d\lambda - 4\pi^2 a \int_0^a r dr \int_0^\infty e^{-\lambda z_2} J_0(\lambda r) J_1(\lambda a) d\lambda\right\}. \tag{4}$$

Now we have

$$\int_0^a r J_0(\lambda r) dr = -\frac{1}{\lambda^2}\int_0^{\lambda a} \xi J_0(\xi) d\xi = \frac{1}{\lambda^2}\left[\xi J_1(\xi)\right]_0^{\lambda a} = \frac{a}{\lambda} J_1(\lambda a),$$

so that, changing the order of integration, we obtain for the flux through the circular section

$$4\pi^2 a^2 mI\left\{1 - \int_0^\infty \frac{e^{-\lambda z_1}}{\lambda} J_1^2(\lambda a) d\lambda - \int_0^\infty \frac{e^{-\lambda z_2}}{\lambda} J_1^2(\lambda a) d\lambda\right\}. \tag{5}$$

Since the length $dz$ carries the current $mIdz$, multiplying this by the flux just found and integrating for the whole length $l$ we obtain for the self-inductance $L$

$$(6) \qquad LI^2 = 4\pi^2 a^2 m^2 I^2 \left\{ l - 2\int_0^l dz \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_1^2(\lambda a) d\lambda \right\}.$$

We may now interchange the order of integration, performing the integration according to $z$,

$$(7) \qquad L = 4\pi^2 a^2 m^2 \left\{ l + 2\int_0^\infty (e^{-\lambda l} - 1) \frac{J_1^2(\lambda a)}{\lambda^2} d\lambda \right\}.$$

We may arrive more quickly at the result (7) by starting from the fundamental definition of the mutual inductance $M_{xx'}$ of two circular lines situated at distances $x$ and $x'$ from one end of the cylinder, both of which are made to occupy all positions possible, when

$$(8) \qquad L = \int_0^l \int_0^l M_{xx'} dx dx',$$

$$(9) \qquad \begin{aligned} M_{xx'} &= \int\int \frac{ds ds' \cos \epsilon}{r} \\ &= 2\pi a a' \int_0^{2\pi} \frac{\cos \theta d\theta}{\sqrt{(x - x')^2 + a^2 + a'^2 - 2aa' \cos \theta}}. \end{aligned}$$

In order to transform the elliptic integral (9) we make use of the integral of Lipschitz (Gray and Mathews, page 72)

$$(10) \qquad \int_0^\infty e^{-\lambda z} J_0(\lambda \omega) d\lambda = \frac{1}{\sqrt{z^2 + \omega^2}},$$

and the addition formula of Neumann (Gray and Mathews, page 27, 69′)

$$(11) \quad J_0(\lambda \omega) = J_0(\lambda a) J_0(\lambda a') + 2 \sum_{n=1}^{n=\infty} J_n(\lambda a) J_n(\lambda a') \cos n\theta,$$

where

$$\omega^2 = a^2 + a'^2 - 2aa' \cos \theta,$$

which being multiplied by $\cos \theta$ and integrated from 0 to $2\pi$ gives

$$(12) \qquad \int_0^{2\pi} J_0(\lambda \omega) d\theta = 2\pi J_1(\lambda a) J_1(\lambda a').$$

Treating (10) in the same manner, and using this result, we convert (9) into

$$(13) \qquad M_{xx'} = 4\pi^2 aa' \int_0^\infty e^{-\lambda|x-x'|} J_1(\lambda a) J_1(\lambda a') d\lambda.$$

We may now perform the integration with regard to $x'$ and then $x$, taking care to notice that the cofactor of $-\lambda$ in the exponential must be positive, so that we put $x - x' = z$ when $x > x'$ and $x - x' = -z$ when $x < x'$, giving

$$\int_0^l M_{xx'} dx'$$

$$= 4\pi^2 aa' \int_0^\infty J_1(\lambda a) J_1(\lambda a') d\lambda \left[ \int_0^x e^{-\lambda z} dz + \int_0^{l-x} e^{-\lambda z} dz \right]$$

$$[\cdots] = \frac{1}{\lambda}(2 - e^{-\lambda x} - e^{-\lambda(l-x)}), \quad \int_0^l [\cdots] dx = \frac{2}{\lambda}\left(l + \frac{e^{-l\lambda} - 1}{\lambda}\right).$$

$$(14) \qquad \int_0^l \int_0^l M_{xx'} dx dx' =$$

$$8\pi^2 aa' \int_0^\infty \left( \frac{l}{\lambda} + \frac{e^{-\lambda l} - 1}{\lambda^2} \right) J_1(\lambda a) J_1(\lambda a') d\lambda.$$

This formula serves for the mutual inductance of two solenoids of the same length but of different diameters, but for our purpose $a = a'$, so that we again obtain (7), since

$$\int_0^\infty \frac{J_1^2(t)}{t} dt = \frac{1}{2}.$$

Let us now introduce the variable $t = \lambda a$ and the constant $z = l/a$ for the ratio of the length to the radius of the solenoid, so that

$$(15) \quad L = 8\pi^2 a^3 m^2 \left[ \frac{z}{2} - \int_0^\infty \frac{J_1^2(t)}{t^2} dt + \int_0^\infty e^{-zt} \frac{J_1^2(t)}{t^2} dt \right].$$

Of these two definite integrals, the first is a constant, whose value we find by the formula given by Nielsen, Handbuch der Theorie der Cylinderfunktionen, page 194, (15) to be $4/3\pi$, while the second is a function of $z$ which we will develop into a power series.

$$(16) \qquad Z = \int_0^\infty e^{-zt} \frac{J_1^2(t)}{t^2} dt.$$

Now we have (Nielsen, page 20, (4) ),

$$(17)\quad J_1^2(t) = \sum_0^\infty \frac{(-1)^s(2s+2)!}{s!\,(s+1)!\,(s+1)!\,(s+2)!}\left(\frac{t}{2}\right)^{2s+2},$$

so that

$$(18)\quad Z = \sum_0^\infty \frac{(-1)^s(2s+2)!}{2^{2s+2}s!\,(s+1)!\,(s+1)!\,(s+2)!}\int_0^\infty e^{-zt}t^{2s}dt.$$

The definite integral is now a gamma function, whose value is easily seen to be

$$(2s)!/z^{2s+1}.$$

Thus we obtain

$$(19)\quad Z = \sum_0^\infty \frac{(-1)^s(2s)!\,(2s+2)!}{2^{2s+2}s!\,(s+1)!\,(s+1)!\,(s+2)!}\,\frac{1}{z^{2s+1}},$$

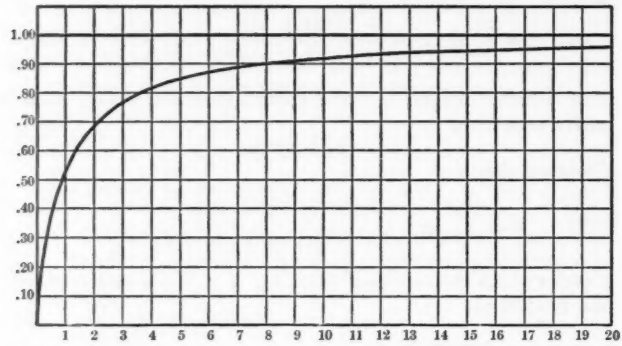
a series which is convergent when $z > 2$. We have accordingly the final result

$$(20)\quad \begin{aligned} L &= 4\pi^2a^3m^2\left\{z - \frac{8}{3\pi}\right. \\ &\qquad \left. + \sum_0^\infty \frac{(-1)^s(2s)!\,(s+2)!}{2^{2s+1}s!\,(s+1)!\,(s+1)!\,(s+2)!}\,\frac{1}{z^{2s+1}}\right\} \\ &= 4\pi^2a^3m^2\left\{z - \frac{8}{3\pi} + \frac{1}{z} - \frac{1}{2z^3} + \frac{5}{8z^5} - \frac{35}{32z^7} + \frac{147}{64z^9}\right. \\ &\qquad \left. - \frac{693}{128z^{11}} + \frac{14{,}157}{1{,}024z^{13}} - \cdots\right\}, \end{aligned}$$

giving the inductance directly as a power series in terms of the ratio of length to radius, when the length of the cylinder is greater than its diameter. The series does not converge rapidly until the length is several times the diameter, but Coffin's formula

$$(21)\quad \begin{aligned} L &= 4\pi an^2\left\{(N-\tfrac{1}{2}) + \frac{z^2}{32}(N+\tfrac{1}{4}) - \frac{z^4}{1{,}024}(N-\tfrac{2}{3})\right. \\ &\qquad \left. + \frac{10z^6}{131{,}072}(N-\tfrac{109}{120}) - \frac{35z^8}{4{,}194{,}304}(N-\tfrac{4321}{4200})\cdots\right\}, \\ &\qquad \left(N = \log_e\frac{8a}{l},\ n = lm\right), \end{aligned}$$

converges better the shorter the cylinder, so that by one of these formulas the calculation may always be made. Coffin has also given a formula by elliptic integrals, which is convenient when tables are at hand. Mr. Gordon Fulcher has




calculated by the three methods values for some twenty ratios, from which he has constructed the annexed graph of $L/4\pi^2a^3m^2z^2$, that is the factor of correction for the ends of the solenoid. When the length is ten times the diameter, five terms of (20) give nine figures of the result.

CLARK UNIVERSITY,
*July*, 1907.

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## ON THE APSIDAL ANGLE IN CENTRAL ORBITS.

BY DR. F. L. GRIFFIN.

(Read in part before the American Mathematical Society, April 27, 1907.)

THERE are two well known laws of central force all of whose trajectories have the same apsidal angle, whatever be the apsidal values of the radius vector, viz., that of Newton and the law that the force varies directly as the distance. For both of these laws the orbits are all conic sections, the apsidal angle being $\pi$ in the former case, and $\frac{1}{2}\pi$ in the latter. Generally, however, the apsidal angle depends upon the apsidal values of the radius vector.

In this paper are considered only those laws of central force for which the force is a function of the distance, having a finite

derivative except at a finite number of distances in any interval. For all such laws, and for all orbits lying wholly within any interval throughout which this derivative is finite, four theorems will be proved. If, for a given law, the derivative is finite for all values of the distance, the theorems hold for all the trajectories. The criteria obtained find immediately two applications: (1) To orbits in the equatorial planes of attracting spheroids—incidentally explaining a well known phenomenon in the motion of the fifth satellite of Jupiter; and (2) to the trajectories for various simple laws of force.

Let $u$ denote the reciprocal of the radius vector, and $u^2P(u)$, the force; and let $P'(u) = d[P(u)]/du$. Evidently, the derivative of the force with respect to the distance is finite if, and only if, $P'(u)$ is finite.

THEOREM I. *In order that the apsidal angle be greater than* $\pi$ (equal to $\pi$, or less than $\pi$) *in every trajectory for a given law, it is necessary and sufficient that* $P'(u)$ *be* (respectively) *everywhere positive* (zero, or negative).

The proof is got by considering the definite integral which gives the apsidal angle. If $h$ be the constant of areas and $\theta$ the longitude, the differential equation of any orbit under the given law is

$$h^2u^2(u + d^2u/d\theta^2) = u^2P(u) \equiv \tfrac{1}{2}u^2F'(u).$$

[$P(u)$, since it has a derivative, is continuous and therefore integrable.] After integration, one has

$$h^2[u^2 + (du/d\theta)^2] = c + F(u),$$

$c$ being a constant. Or, if $\alpha$ and $\beta$ [$\beta > \alpha$] be the apsidal values of $u$,

$$\left(\frac{du}{d\theta}\right)^2 = (\beta^2 - u^2) + \frac{1}{h^2}[F(u) - F(\beta)] \tag{1}$$
$$= (\alpha^2 - u^2) + \frac{1}{h^2}[F(u) - F(\alpha)].$$

Hence the apsidal angle is given by

$$\Theta = \int_\alpha^\beta \frac{du}{\sqrt{\beta^2 - u^2 + \frac{1}{h^2}[F(u) - F(\beta)]}}. \tag{2}$$

*Sufficiency of the condition on $P'(u)$.* Compare $\Theta$ with $\pi$, the apsidal angle in the case of Newton's law, where $P(u) = N$, a constant. For the latter law the second member of (1) is $(\beta - u)[\beta + u - 2N/h^2]$. Hence, if $N$ be selected,

$$N = \tfrac{1}{2}h^2(\alpha + \beta),$$

the Keplerian ellipse will have the same constant of areas and the same apsidal values of $u$ as the original orbit. Therefore

$$\pi = \int_\alpha^\beta \frac{du}{\sqrt{\beta^2 - u^2 + \frac{2N}{h^2}(u - \beta)}}, \tag{3}$$

$$\Theta - \pi = \int_\alpha^\beta \left\{ \frac{1}{\sqrt{\beta^2 - u^2 + \frac{1}{h^2}[F(u) - F(\beta)]}} - \frac{1}{\sqrt{\beta^2 - u^2 + \frac{2N}{h^2}(u - \beta)}} \right\} du. \tag{4}$$

Consider now the function

$$h^2\left\{\beta^2 - u^2 + \frac{2N}{h^2}(u - \beta)\right\} - h^2\left\{\beta^2 - u^2 + \frac{1}{h^2}[F(u) - F(\beta)]\right\},$$

or

$$\Phi(u) \equiv 2N(u - \beta) - F(u) + F(\beta). \tag{5}$$

Since in both orbits $du/d\theta$ vanishes at $u = \alpha$, and at $u = \beta$, it follows that

$$\Phi(\alpha) = \Phi(\beta) = 0. \tag{6}$$

Also,

$$\Phi''(u) = -F'''(u) = -2P'(u).$$

Suppose now that $P'(u) > 0$ for $\alpha \leqq u \leqq \beta$. Then $\Phi''(u) < 0$, and consequently $\Phi'(u)$ decreases throughout the interval $(\alpha, \beta)$. Now since $\Phi'(u)$ has a derivative it is continuous, and therefore vanishes at most once for $\alpha \leqq u \leqq \beta$. But, from (6), $\Phi'(u)$ must vanish once, by Rolle's theorem. Since $\Phi'(u)$ decreases from $u = \alpha$ to $u = \beta$ and vanishes between $\alpha$ and

$\beta$, $\Phi'(\alpha) > 0$ and $\Phi'(\beta) < 0$. Consequently $\Phi(u) > 0$ for $\alpha < u < \beta$.

This shows that the integrand (4) is everywhere positive, whence $\Theta - \pi > 0$ or $\Theta > \pi$.

Similarly, suppose $P'(u)$ everywhere negative; then $\Phi(u) < 0$, the integrand is everywhere negative, and $\Theta < \pi$. Likewise for $P'(u) = 0$, the integrand is zero throughout, and $\Theta = \pi$. Thus, the sufficiency of the condition is established.

*Necessity of the condition on $P'(u)$.* Suppose $P'(u)$ changes sign within the interval $(\alpha, \beta)$. Then within some interval $(\alpha_1, \beta_1)$, $P'(u) > 0$; and within some interval $(\alpha_2, \beta_2)$, $P'(u) < 0$. Hence, by the sufficiency proof, all orbits lying wholly within $(\alpha_1, \beta_1)$ have their apsidal angles greater than $\pi$; and all those lying within $(\alpha_2, \beta_2)$ have their apsidal angles less than $\pi$. But this does not fulfill the requirement that $\Theta$ be greater than $\pi$ (equal to $\pi$, or less than $\pi$) for *all* orbits lying within $(\alpha, \beta)$. Hence the condition is necessary.

THEOREM II. *If $P'(u)$ is positive, and increases with $u$ or is constant, then, in those families of orbits one of whose apsidal distances is constant, the apsidal angle increases as the second apsidal distance decreases.*

[For orbits having a common pericentral distance this means that the apsidal angle is greater the more nearly circular the orbit; for orbits having a common apocentral distance the reverse is true.]

*Case* I. Let $1/\beta$ be the common pericentral distance of two orbits, whose apocentral distances are $1/\alpha_1$ and $1/\alpha_2$, and whose constants of areas are $h_1$ and $h_2$, $h_2 > h_1$. Now, from the differential equations (1) of the two orbits, it follows that for equal values of $u_1$ and $u_2$

$$h_2^2 \left(\frac{du}{d\theta}\right)_2^2 = h_1^2 \left(\frac{du}{d\theta}\right)_1^2 + (h_2^2 - h_1^2)(\beta^2 - u^2). \tag{7}$$

And, since $(du/d\theta)_1$ is real for $\beta \geqq u_1 \geqq \alpha_1$, the same is true of $(du/d\theta)_2$, and moreover $(du/d\theta)_2^2 > 0$ at $u_2 = \alpha_1$. Hence the interval $(\alpha_1, \beta)$ is one of real motion for orbit 2; and $\alpha_2$ lies outside the interval, so that $\alpha_2 < \alpha_1$. The apsidal angles $\Theta_1$ and $\Theta_2$ are given by

$$\Theta_i = \int_{\alpha_i}^{\beta} \frac{du_i}{\sqrt{\beta^2 - u_i^2 + \frac{1}{h_i^2}[F(u_i) - F(\beta)]}} \qquad (i = 1, 2). \tag{8}$$

To compare these two angles, make the following linear substitutions: $u_1 = a\beta + bu$, $u_2 = u$, where

$$a = \frac{\alpha_1 - \alpha_2}{\beta - \alpha_2}, \quad b = \frac{\beta - \alpha_1}{\beta - \alpha_2} = 1 - a.$$

Both $a$ and $b$ are evidently positive and less than unity; and $u_1 > u$ if $\beta > u$. Then $\Theta_1 - \Theta_2$ becomes

$$\Theta_1 - \Theta_2 = \int_{\alpha_2}^{\beta} \left\{ \frac{b}{\sqrt{\beta^2 - (a\beta + bu)^2 + \frac{1}{h_1^2}[F(a\beta + bu) - F(\beta)]}} - \frac{1}{\sqrt{\beta^2 - u^2 + \frac{1}{h_2^2}[F(u) - F(\beta)]}} \right\} du, \tag{9}$$

for $u = \alpha_2$ at $u_1 = \alpha_1$. A function similar to that used in proving Theorem I will be defined here,

$$\Psi(u) \equiv h_1^2 b^2 [h_2^2(\beta^2 - u^2) + F(u) - F(\beta)] - h_2^2 [h_1^2(\beta^2 - (a\beta + bu)^2) + F(a\beta + bu) - F(\beta)]. \tag{10}$$

Since in both orbits $du/d\theta$ vanishes for $u = \alpha_2$ and for $u = \beta$,

$$\Psi(\alpha_2) = \Psi(\beta) = 0. \tag{11}$$

Evidently

$$b^{-2}\Psi''(u) = h_1^2 F''(u) - h_2^2 F''(a\beta + bu) = 2h_1^2 P'(u) - 2h_2^2 P'(a\beta + bu)$$

or

$$\tfrac{1}{2} b^{-2}\Psi''(u) = (h_1^2 - h_2^2) P'(u) - h_2^2 [P'(u_1) - P'(u)]. \tag{12}$$

Since $h_2 > h_1$ and $P'(u)$ is positive, the first term on the right side of (12) is negative, and since $P'(u)$ increases with $u$ or is constant, and $u_1 > u$ for $\beta > u$, the second term is negative or zero. Hence $\Psi''(u) < 0$ for $u < \beta$, and $\Psi'(u)$ decreases from $u = \alpha_2$ to $u = \beta$, vanishing at most once, because continuous. But, from (11), it vanishes once; this shows that $\Psi'(\alpha_2) > 0$ and $\Psi'(\beta) < 0$. Consequently $\Psi(u) > 0$ for $\beta > u > \alpha_2$, and the integrand is positive throughout, showing that $\Theta_1 - \Theta_2 > 0$.

Now it was shown that $h_2 > h_1$ implies $\alpha_2 < \alpha_1$; also it is obvious that $h_2 = h_1$ implies $\alpha_2 = \alpha_1$. It follows, therefore, that the greater the value of $\alpha$ — that is, the more nearly circular the orbit — the greater is the apsidal angle. The theorem is thus established for families of orbits with the same pericentral distance; it remains to examine those having the apocentral distance constant.

*Case* II. Let $1/\alpha$ be the common apocentral distance of two orbits whose pericentral distances are $1/\beta_1$ and $1/\beta_2$, and whose constants of areas are $h_1$ and $h_2$, $h_2 > h_1$. An examination of the differential equations corresponding to (1) shows that for equal values of $u_1$ and $u_2$

$$h_1^2\left(\frac{du}{d\theta}\right)_1^2 = h_2^2\left(\frac{du}{d\theta}\right)_2^2 + (h_1^2 - h_2^2)(\alpha^2 - u^2). \tag{13}$$

And, since $(du/d\theta)_2$ is real for $\beta_2 \geqq u_2 \geqq \alpha$, the same is true of $(du/d\theta)_1$, and moreover $(du/d\theta)_1^2 > 0$ at $u_1 = \beta_2$. Hence the orbit 1 is real throughout the interval $(\alpha, \beta_2)$, and beyond. Therefore $\beta_1 > \beta_2$. The difference of the apsidal angles may then be written

$$\Theta_1 - \Theta_2 = \int_\alpha^{\beta_1} \left\{ \frac{1}{\sqrt{\alpha^2 - u^2 + \frac{1}{h_1^2}[F(u) - F(\alpha)]}} - \frac{f}{\sqrt{\alpha^2 - (e\alpha + fu)^2 + \frac{1}{h_2^2}[F(e\alpha + fu) - F(\alpha)]}} \right\} du, \tag{14}$$

where

$$e = \frac{\beta_1 - \beta_2}{\beta_1 - \alpha}, \quad f = \frac{\beta_2 - \alpha}{\beta_1 - \alpha} = 1 - e.$$

Evidently both $e$ and $f$ are positive and less than unity; and $e\alpha + fu < u$ for $\alpha < u$. Let

$$\begin{aligned}\chi(u) \equiv h_1^2[h_2^2(\alpha^2 - (e\alpha + fu)^2) + F(e\alpha + fu) - F(\alpha)] \\ - f^2 h_2^2[h_1^2(\alpha^2 - u^2) + F(u) - F(\alpha)].\end{aligned} \tag{15}$$

Then, for the same reason as in (11),

$$\chi(\alpha) = \chi(\beta_1) = 0; \tag{16}$$

and

$$(17)\quad \frac{1}{2f^2}\chi''(u) = (h_1^2 - h_2^2)P'(u) - h_1^2[P'(u) - P'(e\alpha + fu)].$$

Since both terms of the right member of (17) are negative for $u > \alpha$, $\chi''(u) < 0$ so that the discussion of $\Psi(u)$ in Case I applies equally to $\chi(u)$ here. Hence, $\Theta_1 - \Theta_2 > 0$, or $\Theta_1 > \Theta_2$. Finally, then, it follows that the smaller the pericentral distance — that is, the less nearly circular the orbit — the greater the apsidal angle.

COROLLARY. *If two orbits have no apsidal distance in common, both apsidal distances in the first being less than the corresponding distances in the second, the apsidal angle in the first is greater than that in the second.*

For, let the apocentral and pericentral distances in orbit 1 be, respectively, $1/\alpha_1$ and $1/\beta_1$, and in orbit 2, $1/\alpha_2$ and $1/\beta_2$. Consider a new orbit — orbit 3 — with pericentral distance $1/\beta_1$ and apocentral distance $1/\alpha_2$. Then, by Case I, $\Theta_1 > \Theta_3$; and by Case II, $\Theta_3 > \Theta_2$. Hence, $\Theta_1 > \Theta_2$.

THEOREM III. *If $P'(u)$ is negative, and a decreasing function of $u$ or constant, then in those families of orbits one of whose apsidal distances is constant, the apsidal angle increases with the other apsidal distance.*

The proof is got as in Theorem II, except that $\Psi''(u)$ and $\chi''(u)$ are both positive, which reverses the sign of the integrand in each case. Hence the conclusion. The reverse of the above corollary holds true here.

[Remark. Theorems II and III might have been anticipated intuitionally from the fact that the difference between the variations of the given law and that of Newton is most marked near the center of force. One should therefore expect the apsidal angle to differ from $\pi$ by larger amounts, the nearer the center the orbit extends.]

THEOREM IV. *About the center of force there exists a system of concentric spheres, such that those orbits lying wholly between the surfaces of consecutive spheres have their apsidal angles all greater than $\pi$, or else all less than $\pi$.*

[The hypothesis upon $P(u)$, viz., $P'(u)$ finite save at a finite number of points in any interval, is, however, to be strengthened: $P(u)$ has only a finite number of maxima and minima in any interval.]

By reason of these two hypotheses there exist, in any inter-

val, only a finite number of values of $r$ for which $P'(u) = 0$, or $P'(u) = \infty$. Select these values as radii of concentric spheres. Then, between consecutive surfaces, $P'(u)$ neither vanishes nor becomes infinite, and consequently keeps the same sign. Hence, the difference $\Theta - \pi$ has the same sign for all orbits in the region specified.

## Applications of Theorems.

While all of the above theorems will be utilized in what follows, Theorem I, by reason of its simplicity and generality, offers the most powerful criterion. As will be apparent in the second class of applications to be made, an inspection of the law of force is all that is necessary to ascertain whether the apsidal angle is greater than $\pi$ in some orbits, and less than $\pi$ in others, or whether the difference $\Theta - \pi$ must always have the same sign.

*Class I*: *Motion in the equatorial plane of a spheroid.*

*Case I. The oblate spheroid.* The resultant force varies, not inversely as the square of the distance, but in a much more complicated manner. Denoting by $e$ the eccentricity and by $a$ the major semiaxis of a meridian section, the components of attraction, $X$ and $Y$, parallel to rectangular axes in the equatorial plane are given by *

$$(18)\qquad \frac{X}{x} = \frac{Y}{y} = -\tfrac{3}{2}\frac{k^2 M \cdot m}{(ae)^3}\left[\arcsin\frac{ae}{\sqrt{x^2+y^2}} - \frac{ae}{\sqrt{x^2+y^2}}\sqrt{1-\frac{a^2e^2}{x^2+y^2}}\right],$$

$m$ being the mass of the attracted particle, $M$ that of the spheroid, and $x$ and $y$ the coordinates of the particle, the origin being at the center of the spheroid. Now (18) shows that the force is central; and, introducing the variable $u$, it is found that

$$(19)\qquad P(u) = \frac{K}{u^3}\left[\arcsin aeu - aeu\sqrt{1-(aeu)^2}\right],$$

where $K$ is a positive constant. The right member of (19) is

* Moulton, F. R. Introduction to celestial mechanics, pp. 123; 118.

representable by a power series in $aeu$, convergent for all points exterior to the spheroid; thus

$$P(u) = \sum_{n=0}^{n=\infty} A_n (aeu)^{2n}, \tag{20}$$

*where the $A_n$ are all positive.* Evidently, then, for all exterior points, $P'(u)$ is positive and increases with $u$; consequently Theorems I and II apply.

Hence, *every orbit described in the equatorial plane of an attracting oblate spheroid has its apsidal angle greater than $\pi$; and, the nearer the center of force the orbit lies, the greater the apsidal angle.*

This finds an immediate application in the motion of the fifth satellite of Jupiter, whose orbit deviates exceedingly little from the equatorial plane of the planet. The oblateness of Jupiter is considerable, so that $P(u)$ differs from the newtonian term in the series (20) very markedly for large values of $u$. This, with the fact that the satellite's orbit is very near the planet, should explain the large difference between its apsidal angle and $\pi$, amounting in a year to about 900°.

This "advance of the line of apsides" has also been fully explained from the standpoint of perturbations,* the orbit being regarded as an ellipse whose elements vary; the difference between the apsidal angle and $\pi$ being considered as a forward rotation of the major axis.

*Case II. The prolate spheroid.* The attraction upon a particle in the equatorial plane is given by †

$$P(u) = \frac{K}{u^3}\left[aeu\sqrt{1+(aeu)^2} - \log\left(aeu + \sqrt{1+(aeu)^2}\right)\right], \tag{21}$$

$K$ being a positive constant, and $a$ and $e$ having the same meanings as in Case I. The right member of (21) is developable as a power series in $aeu$,

$$P(u) = \sum_{n=0}^{n=\infty} B_n (aeu)^{2n}, \tag{22}$$

where

$$B_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{(-1)^n}{2n+3}.$$

* Moulton, loc. cit., pp. 232–233.
† Moulton, loc. cit., p. 123.

This series is convergent for $aeu < 1$; whether this includes all exterior points depends upon the value of $e$.

Since the first coefficient of $P'(u)$ is negative, it is evident that $P'(u) < 0$ for sufficiently small values of $u$. Hence all orbits lying outside of a certain circle have their apsidal angles less than $\pi$.

Again, since the ratio of consecutive coefficients in $P''(u)$ is

$$\frac{(2n+1)^2(2n+3)}{(2n-1)(2n)(2n+5)}(-1),$$

which has its greatest numerical value, $\frac{4}{14}\frac{5}{4}$, for $n = 1$, it follows that for $aeu < \sqrt{\frac{14}{45}}$, if the terms of $P''(u)$ be grouped in pairs, each pair will have a negative sum. So that, for $u$ sufficiently small, $P'(u)$ is a decreasing function; therefore the apsidal angle decreases toward the center of force.

Finally, calling $aeu = \lambda$,

$$P'\left(\frac{\lambda}{ae}\right) = \frac{3K(ae)^3}{\lambda^4\sqrt{1+\lambda^2}}\left[\log(\lambda+\sqrt{\lambda^2+1}) - (\lambda + \tfrac{1}{3}\lambda^3)\right],$$

which can be shown to be everywhere negative. Thus, let $\log(\lambda + \sqrt{\lambda^2+1}) - (\lambda + \frac{1}{3}\lambda^3) = \Phi(\lambda)$. Then

$$\Phi(0) = 0,\ \Phi'(\lambda) = \frac{1}{\sqrt{1+\lambda^2}} - (1+\lambda^2).$$

Since $\Phi'(\lambda) < 0$ for every real value of $\lambda$, $\Phi(\lambda) < 0$ for $\lambda > 0$. Hence, *the apsidal angle is less than $\pi$ in all the trajectories.*

*Class II. Orbits for various simple laws.*

The criterion of Theorem I yields especially simple results when applied to the following types of laws of force:

*Case I. Force varies inversely as the $n^{th}$ power of the distance.* If

$$u^2P(u) = k^2\frac{1}{r^n}, \quad P(u) = k^2u^{n-2},$$

and $P'(u) = (n-2)k^2u^{n-3}$. Three cases arise:

(1) $n = 2$; Newton's law, $\Theta = \pi$.

(2) $n > 2$; $P'(u) > 0$, $P''(u) \geqq 0$. Hence, theorems I and II both apply. $\Theta > \pi$, and increases toward the center.

(3) $n < 2$; $P'(u) < 0$, $P''(u) > 0$. Theorem I applies, but not Theorem III. $\Theta < \pi$.

In many of the cases included here it is quite impossible to evaluate in closed form the definite integral defining $\Theta$. Thus, under (2) are included the well known "inverse fifth power," which leads to elliptic integrals; also, all the inverse higher powers leading to abelian and hyperabelian integrals. By Theorem I all these integrals are greater than $\pi$ and increase with the apsidal values of $u$.

Under (3) are included: The "inverse first power," which leads to logarithms, and which is usually omitted from the treatises; the simple law of the "direct first power," where $\Theta = \frac{1}{2}\pi$; the case of the constant force, leading to an elliptic integral, and the higher direct powers with their involved integrals. These apsidal angles are, however, by Theorem I, all less than $\pi$.

*Case* II. *Force varies as log* $1/r^m$ *and inversely as* $r^n$. Here, since

$$u^2P(u) = (k^2m/r^n)\log u, P'(u) = k^2m[(n-2)\log u \cdot u^{n-3} + u^{n-3}],$$

or

$$P'(u) = k^2mu^{n-3}[1 + (n-2)\log u]. \tag{23}$$

From (23) it is evident that for $n = 2$, $P'(u) > 0$ for $u > 0$; for $n > 2$, $P'(u) \gtrless 0$ according as $\log u \gtrless -1/(n-2)$, and for $n < 2$, $P'(u) \gtrless 0$, according as $\log u \lesseqgtr 1/(2-n)$. That is, if $n > 2$, $\Theta > \pi$ in orbits within a certain sphere; if $n = 2$, $\Theta > \pi$ in all orbits; if $n < 2$, $\Theta > \pi$ in orbits without a certain sphere. The opposite inequalities hold in other parts of space.

*Case* III. *Force varies as* $e^{\phi(u)}$ *and inversely as* $r^n$. In this case, since $P'(u) = k^2u^{n-2}[\phi'(u) + (n-2)u]e^{\phi(u)}$, it follows that if $\phi(u)$ be everywhere real, $P'(u)$ has the same sign as $\phi'(u) + (n-2)u$. While a discussion of the cases arising when $\phi(u)$ is arbitrary is of small importance, the following special cases may well be noted. If $\phi'(u) > 0$ for $u > 0$, the inequalities of Case II hold; if $\phi'(u)$ is everywhere negative, those inequalities are to be reversed.

The laws of force which have been examined, while only a few, embrace most of those commonly considered in treatments of central force problems, and the simplicity of the criterion in these cases encourages the hope that it may be found useful elsewhere.

WILLIAMS COLLEGE,
*May*, 1907.

# THE MAXIMUM VALUE OF A DETERMINANT.

BY PROFESSOR ELLERY W. DAVIS.

HADAMARD* has shown that the maximum value reached by the modulus of a determinant of order $n$ the moduli of whose elements do not exceed unity is $n^{n/2}$. The result is fundamental in Fredholm's theory of integral equations.†

The problem, when the elements are required to be real, was studied by Professor Hathaway and myself in 1882. I do not think Mr. Hathaway's work was published. My own was published‡ only in the abstract here reproduced.

"The elements of a determinant being restricted to a variation between the limits $-a$ and $+a$, it is found that for all determinants whose order is greater than 2, a numerical maximum will be obtained by putting the elements $-a$ in the principal diagonal and making all the other elements of the determinant $+a$. If we denote such a determinant of order $n$ when $a=1$ by $D_n$ and the minor of an element in the $i$th row and $k$th column by $A_{ik}$, we have always $A_{ii}=D_{n-1}$, $A_{ik}=-D_{n-1}/(n-3)$, so that

$$D_n=-\left(1+\frac{n-1}{n-3}\right)\cdot D_{n-1}=+\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)D_{n-2}$$

$$=\pm\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)\cdots\left(1+\frac{5}{3}\right)\left(1+\frac{4}{2}\right)\left(1+\frac{3}{1}\right)D_3$$

$$=\pm\,(n-2)2^{n-1}, \text{ since } D_3=4.$$

"The sign will be $\pm$ according as $n$ is $\overset{\text{odd}}{\text{even}}$. The effect of a change in any element is to lessen the greatest factor leaving the rest the same.

"For the maximum cubic determinant $D_n^{(3)}a^n$, we have merely to make all the strata identical and equal to $D_n^{(2)}$. Its value is $\pm n!\,D_n^{(2)}a^n$.

"The four-dimensional determinant may be looked upon as a sort of determinant of plane determinants, the terms of the devel-

* *Bulletin des sciences math.*, 1893, p. 240. Pascal, I Determinanti, p. 240.

† *Acta Math.*, vol. 27 (1903), p. 365.

‡ Johns Hopkins University Circular, vol. 2, No. 20 (December, 1882), p. 22.

opment being cubic determinants of which the plane determinants are the strata. In this quasi-determinant the same rule of signs applies as in the ordinary determinant and we get for the maximum value accordingly

$$\pm D_n^{(2)} n!\, D_n^{(2)} a^n = \pm n!\,(D_n^2)^2.$$

Similarly

$$D_n^{(2p)} = \pm (n!)^{p-1} (D_n^{(2)})^p, \qquad D_n^{(2p+1)} = \pm (n!\, D_n^{(2)})^p.$$

"In any of these determinants we have at once a formula for the maximum number of positive terms in the development. This is for $D_n^{(q)}$

$$\frac{(n!)^{q-1} \pm D_n^{(q)}}{2}.$$

"We use the $\pm$ sign according as $D_n^{(q)}$ is $\pm$."

A simple example serves to show that the limit reached when the elements are real may be exceeded when the elements are complex. Thus

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4,$$

but

$$\begin{vmatrix} i & 1 & 1 \\ 1 & \dfrac{-1+i}{\sqrt{2}} & 1 \\ 1 & \dfrac{1+i}{\sqrt{2}} & -1 \end{vmatrix} = 2(\sqrt{2}+1).$$

This explains why Hadamard's limit exceeds mine.

UNIVERSITY OF NEBRASKA,
*July*, 1907.

# THE INVARIANT SUBSTITUTIONS UNDER A SUBSTITUTION GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

WE begin with the case where the substitution group $G$ is transitive and of degree $n$. If the subgroup $G_1$ composed of all the substitutions of $G$ which omit a given letter is of degree $n-1$, there is no substitution which involves any of the letters contained in the substitutions of $G$ and is also commutative with every substitution of $G$. In considering substitutions which are commutative with every substitution of $G$ we shall confine ourselves to those which involve no letters that are not also contained in $G$, since every substitution which does not involve any of the letters of $G$ is clearly commutative with every substitution of $G$. Hence we may say: *when the degree of $G_1$ is $n-1$, identity is the only substitution which is commutative with every substitution of $G$.*

When $G_1$ is identity it is well known that there are exactly $n$ substitutions which are commutative with every substitution of $G$, and that these constitute a group which is simply isomorphic with $G$, known as the *associate* of $G$, whenever $G$ is non-abelian.* It remains to consider the case where $G_1$ is of degree $n-\alpha$ $(1<\alpha<n)$. All the substitutions which transform $G_1$ into itself constitute a subgroup of order $\alpha g \div n$, $g$ being the order of $G$. This subgroup may clearly be constructed by establishing a $(g_1, 1)$ isomorphism between a group of order $\alpha g \div n$ and a regular group of order $\alpha$, $g_1$ being the order of $G_1$. If this regular group is abelian it is transformed into $n \div \alpha$ distinct regular groups by the substitutions of $G$ and a simple isomorphism between these groups which is so constructed that all the conjugates of a given substitution correspond will be composed of all the substitutions which are commutative with every substitution of $G$. If the regular group of order $\alpha$ is non-abelian its associate is transformed in the manner stated. Hence we have the theorems: *The necessary and sufficient condition that there are substitutions besides identity which are com-*

* *Quar. Jour. of Math.* vol. 28 (1896), p. 249. When $G$ is abelian it is self-associate.

*mutative with every substitution of a transitive group of degree n is that the subgroup which is composed of all its substitutions omitting a given letter is of a degree lower than n — 1. If the degree of this subgroup is n — α, there are exactly α substitutions which are separately commutative with every substitution of the transitive group, where α may be any divisor of n but can have no other value.*

This theorem may be regarded as a generalization of the theorem that there are just $n$ substitutions which are commutative with every substitution of a regular group. It is also directly applicable to intransitive groups, since the total number of substitutions which are commutative with every substitution of such a group and do not interchange any of its systems of intransitivity is composed of the direct product of those which are commutative with all the substitutions of its transitive constituents. While the number of substitutions which are commutative with every substitution of a transitive group is always a divisor of the degree of the group, the number of those which have this property and do not permute the systems is a divisor of the product of the partitions of this degree with respect to addition when the group is intransitive.* Although every transitive group which is such that there are substitutions which are commutative with every one of its substitutions is necessarily imprimitive,† it is not true that every imprimitive group has this property, since there are imprimitive groups of degree $n$ which involve subgroups of degree $n-1$. The preceding developments have close contact with those given on page 124 of Maillet's Thèse de Doctorat, 1892, and by Burnside, "Theory of groups of finite order," 1897, page 217.

The $\alpha$ substitutions which are commutative with every substitution of $G$ when $G_1$ is of degree $n-\alpha$ are not necessarily contained in $G$ whenever $\alpha > 1$. They are however contained in the largest group of degree $n$ which transforms $G$ into itself. We thus arrive at a generalization of a property of the holomorph of a group, which may be stated as follows: The largest group of degree $n$ which transforms a transitive group of degree

* The addends are clearly the degrees of the systems of intransitivity and the total number of substitutions which are commutative with every substitution of an intransitive group is a divisor of the product of the degrees of its transitive constituents and the number of ways in which these constituents may be transformed into each other.

† Except the cyclic group of prime order.

$n$ into itself contains $\alpha$ substitutions which are commutative with every substitution of this transitive group when its subgroup composed of all the substitutions which omit a given letter is of degree $n - \alpha$. In the case of the holomorph of $G$, $\alpha = n$.

UNIVERSITY OF ILLINOIS,
*July*, 1907.

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## SHORTER NOTICES.

*Leçons d'Algèbre et d'Analyse à l'Usage des Elèves des Classes de Mathématiques spéciales.* Par JULES TANNERY. Tome Premier. Paris, Gauthier-Villars, 1906. 423 + vii pp.

THIS text is prepared for students of the classe de mathématiques spéciales of the French lycées, in which boys of 18 or 19 prepare themselves for admission to the Ecoles normale and polytechnique. In so doing, they go over an amount of mathematics which seems overwhelming.* If the contents of the volume under review are even approximately an indication of what is taught successfully in one year, in any one of several subjects pursued by classes of boys of ordinary ability, then we have indeed much to learn from the teachers in the French lycées. But the contrast is less pronounced when we consider the difference in aim. The French adapt their courses to the abilities of the 2 per cent. or 5 per cent. who are the most gifted; Americans adapt their work to the capacity of the average boy. In the French lycées the 95 per cent. or 98 per cent. who fail in the first trial repeat the course during a second year, whereupon about 25 per cent. succeed. Less than half are said to succeed even after the third trial. Much can be said for and much against such a system of highly competitive examinations.

The first chapter of Tannery's book, covering 58 pages, contains a detailed exposition of irrational numbers. Adopting an easy conversational style, the author makes the subject very clear, except that, in a few instances, it is not stated from the start what the underlying assumptions are. Whenever such doubt lurks in the mind of the reader, he will find that the

* BULLETIN, vol. 6 (1900), p. 233.

matter is cleared up later on, when a restatement of the argument is made. Thus, on page 6, the author states in a footnote the assumption, now usually called the assumption of Archimedes, which was tacitly made on page 4. Tannery calls irrational numbers (page 2), "une façon de parler plutôt qu'une réalité." His theory of the irrational, although originally developed independently, resembles that of Dedekind in starting out with the notion of a partition (coupure). In the elegant development of this theory in the volume under review, the author makes ingenious use of a color system. Of the two classes into which a segment incommensurable to the unit segment on a line divides the points standing for rational numbers, he imagines the inferior class to be represented by blue points and the superior class by red points, while the irrational number in question is pictured by a white point. When the partition is effected by a commensurable segment, the points representing rational numbers are divided into three classes, namely the inferior (blue), the superior (red), and the single intermediate (white). This color imagery greatly facilitates the discussion.

Thus, two irrational numbers $A$ and $B$ are said to be equal when the blue and red points of the one are the very same, respectively, as those of the other. The sum (product) of $A$ and $B$ is a number greater than the sum (product) of any two blue numbers belonging, respectively, to $A$ and $B$; this sum (product) is a number less than the sum (product) of any two red numbers.

Chapters II and III are given to polynomials. The real object of algebra is asserted to be the study of polynomials from different points of view. The author explains the graphic representation and the differentiation of polynomials, derives Taylor's theorem by the method of undetermined coefficients, and touches upon the singular points of graphs.

In chapters IV–VII are treated rational fractions, the greatest common divisor, and imaginaries. The resolution into partial fractions is effected by a process unusual in our text-books. Suppose $f(x)$ is a polynomial of lower degree than $\phi(x) \equiv (x-a)^\alpha(x-b)^\beta \cdots (x-l)^\lambda$. Writing $x = a + h$, $\phi_1(x) \equiv (x-b)^\beta \cdots (x-l)^\lambda$, and

$$P_\alpha\left(\frac{1}{x-a}\right) \equiv \frac{A_0}{(x-a)^\alpha} + \frac{A_1}{(x-a)^{\alpha-1}} + \cdots + \frac{A_{\alpha-1}}{x-a},$$

we obtain

$$\frac{f(a+h)}{\phi_1(a+h)} = A_0 + A_1 h + \cdots + A_{\alpha-1}h^{\alpha-1} \frac{h^\alpha R(h)}{\phi_1(a+h)}.$$

Dividing by $h^\alpha$ and then writing $x - a$ for $h$, we have

$$\frac{f(x)}{\phi(x)} = P_\alpha\left(\frac{1}{x-a}\right) + \frac{f_1(x)}{\phi_1(x)}.$$

This process is repeated on $f_1(x)/\phi_1(x)$ with respect to $(x-b)^\beta$, and so on. It is thus made evident in a natural and convincing way that, for any factor $(x-a)^\alpha$ of the denominator, all powers of $x-a$, from 1 to $\alpha$, must in general appear as denominators of partial fractions. The numerators $A_0$, $A_1$ $\cdots$ are determined by the process of division.

In the treatment of imaginaries Tannery is influenced by Cauchy, who looked upon $i$ as a real variable. The "fundamental theorem of algebra" is expressed by Tannery on page 223 in this form: Given a polynomial with real coefficients in two variables $x$ and $i$, then there is a binomial $\alpha + \beta i$ ($\alpha$ and $\beta$ real) such that, on writing $\alpha + i\beta$ for $x$, the result is divisible by $i^2 + 1$. The letter $i$ plays only a very special role; for, the remainder resulting from a division of a polynomial by $i^2 + 1$ is the sole object of consideration. Two polynomials involving $i$ are said to be equal when their remainders are equal, or when their difference is exactly divisible by $i^2 + 1$. Tannery remarks that this definition satisfies the three essential conditions of equality: an object is equal to itself; if a first object is equal to a second, then the second is equal to the first; if two objects are equal to a third, they are equal to each other. It is then evident that a polynomial in $i$ is equal to its remainder, and in particular that the polynomial $i^2$ is equal to $-1$, which is the remainder resulting from the division of $i^2$ by $i^2 + 1$. When no confusion arises, one may regard $i$ either as a real variable or as defined by $i^2 = -1$. From either point of view, two complex numbers $a + a'i$ and $b + b'i$ are equal if $a = b$ and $a' = b'$. The definitions of the fundamental operations on imaginaries are then set forth.

In the graphic representation of imaginaries the author starts out by marking off real and pure imaginary numbers along rectangular axes, without attempting to assign any reason for

avoiding oblique axes. Later a justification for this procedure appears as the result of successive multiplication of the vector 1 by $i$. The chapter ends with a trigonometric treatment of the roots of unity.

The remaining three chapters are on combinations and permutations, the binomial formula, equations of the first degree and determinants. Everywhere we recognize the resolute endeavor to present the subject with perfect sincerity. The author says in the preface: "J' ai horreur d'un enseignement qui n'est pas toujours sincère: le respect de la vérité est la première leçon morale, sinon la seule, qu'on puisse tirer de l'étude des sciences." F. CAJORI.

*Leçons d'Algèbre et d'Analyse.* Par JULES TANNERY. Tome Second. Paris, Gauthier-Villars, 1906. 638 pp.

THIS volume, which is the second of a comprehensive treatise on this field of algebra and analysis, like the other volumes, is prepared for the use of special students of mathematics of the Sorbonne, Paris. The chapter headings are as follows: series, functions of a real variable, series of functions, applications in the study of a function in the separation and calculation of the roots of an equation, algebraic equations, differential notation and plane curves, and notions of integral calculus.

The chapter on series deduces, with a simplicity and clearness that are ample for beginners of collegiate grade, the fundamental notions of the subject, establishes the ordinary tests for convergence and divergence, examines for convergence many series of frequent occurrence, and closes with seven pages of interesting exercises on convergence, divergence, and equivalence of series. All this is done in the compass of 53 pages and that, too, without material omission of essentials.

The meanings of variable, of function, of the phrases "appertaining to an interval" and "lying within an interval," and of bounds are given and exemplified in the next chapter on functions of a real variable. Common geometric notions are first given, then shown to be too indefinite for the purposes of analysis; then the analytic definitions are given for the following: curve, continuity, functionality, and increasing, decreasing, and discontinuous functions. The meaning and domains of validity of inverse, logarithmic, circular, and exponential functions are pointed out. A development of the properties of

hyperbolic functions completes the theoretical part of the chapter. The principles deduced are reviewed and applied by a four-page list of well-selected exercises.

The third chapter renders precise and trustworthy the terminology and basic notions involved in deriving functions of real variables, and gives general methods of deriving, both totally and partially, functions of real variables with real, and with imaginary coefficients. Rolle's theorem and the so-called theorem of finite differences are deduced and four pages of exercises are added. The functions here treated being capable of graphic representation, the author satisfies himself with the degree of rigor which such functions require, and postpones for later consideration the restrictions necessary for absolute analytical rigor. At appropriate places the student is cautioned as to the inadequacy of these first treatments. He is thus led to feel that while he is always getting the truth, and as much of it as he can assimilate, yet it is not the *whole* truth. This procedure cultivates in the student open-mindedness and the feeling that there is still more to learn.

Series whose terms are functions of a variable, series in integral powers of a single variable—power series—first with real, then with imaginary coefficients, and the Taylor and Maclaurin developments are carefully treated and extensively applied to both algebraic and transcendental functions in the next chapter. Several illusory forms are given special attention and their true values are defined.

Then follows a study of variations of functions, of roots, multiple roots, asymptotes, convexity, concavity, points of inflexion, and separation of roots. Rolle's theorem is again taken up, and methods of approximating roots of functions are taught and given extensive application.

The relations of coefficients and of roots of algebraic equations, symmetric functions, Waring's method, methods of elimination and transformation, reciprocal, binomial, cubic, and quartic equations, together with other topics of the theory of algebraic equations, are given full and yet concise consideration. The differential and determinant notations applied to the study of plane curves, with a clearly logical and lucid development of the integral calculus, running through 154 pages, constitute the next chapter, which completes the volume.

From this synopsis it is plain that the French order of procedure—for M. Tannery's order is customary in France—

through the several branches of mathematical analysis differs materially from the order of American texts. Of course, it does not follow that because another nation prefers a certain plan it is the best thing for us to adopt the plan. It must be admitted, however, that the loosely articulated subject matter as given in our so-called college algebra, which closes with a rudimentary and isolated treatment of the theory of equations, then trigonometry, then analytics, then calculus, leaves much to be desired. Many desirable things might be suggested by the close study of such masterly treatments as this of M. Tannery by would-be authors of college algebras and of other mathematical books for use in our schools. Both in point of gradation of inherent difficulties and in point of simplicity or treatment without loss of rigor the French text-book writers can teach us much. The logical order and mode of this volume are admirable.

Again, the plan by which M. Tannery gradually inducts the learner into the logical niceties of the subject, by giving him from time to time enough to satisfy present needs and as much as his stage of mathematical maturity and advance justifies, then reverting to the matter later, when experience has ripened sufficiently to enable the novice to interpret what he is doing, is a distinct advance pedagogically upon the common practice of American college professors. The latter are too prone to seek fulness of detail the first time the subject is taken up. The reviewer would invite their attention to M. Tannery's procedure, for these Leçons were evidently prepared to inform and to educate rather than merely to present a beautifully perfect mathematical exhibit.

Only half a dozen typographical errors, and these of no particular consequence, have been discovered in the entire volume. Typography and general mechanical excellence are sufficiently guaranteed by the imprint of Gauthier-Villars.

G. W. MYERS.

*Mathématiques. Principes et Formules de Trigonométrie Rectiligne et Sphérique.* Par J. PIONCHON, Professeur à la Faculté des Sciences, Directeur de l'Institut Electrotechnique de l'Université de Grenoble. Paris, Gauthier-Villars, et Grenoble, A. Gratier et J. Rey, 1904. 5 francs.

THIS is one of the 70 odd volumes now appearing from the press of Gratier et J. Rey, of Grenoble, under the editorial

supervision of J. Pionchon and bearing the general title Bibliothèque de l'Elève-Ingenieur. This Bibliothèque is to consist of five sections entitled respectively: Mathématiques, Mécanique, Physique industrielle, Electricité industrielle, and Economie industrielle. The collection as a whole is to comprise a recapitulation of the fundamental notions, both theoretical and practical, of the applied sciences. M. Pionchon is to be assisted in this undertaking by about a dozen of the foremost representatives in France of the several subjects to be treated. The book under review is one of the half-dozen numbers of this collection that have recently appeared. A dozen other volumes of the general series are announced to be in preparation.

This little volume of 146 pages is designed for use as a hand-book for practical engineers. It is contended by the author that the needs of the practicing engineer for trigonometry are met neither by the standard texts of trigonometry prepared by professors for their classes, nor by the treatises written by experts for mathematicians. The former are held to be too scanty and the latter too extensive for practice. It is believed therefore that in the bibliography for engineering students there is a real want which this compilation seeks to supply.

This book, therefore, is not intended for a first study of the subject. Nor is it a mere compilation of the formulas needed in engineering practice. It is a sort of running sketch of the substance of the theory of trigonometry, of sufficient fulness to enable one who has once learned the science to readily recover the *rationale* of formulas and transformations. A feature of the book is the large number of such approximate formulas as are useful in practice. Lists of trigonometric integrals and of trigonometric series, full enough to meet the demands of extended practice, are also included. It is perhaps needless to say that special attention has been given to adaptations and simplifications of formulas for the solution of triangles, both plane and spherical.

A rather full appendix, comprising notes on Divers cases of maximum and minimum of certain geometric magnitudes in figures satisfying given conditions, is also included. In this appendix such questions are treated as "To find on a right line the smallest segment which may be seen from a given point under a given angle"; and "Through a point situated on the interior of an angle, to insert between the sides a rectilinear segment of minimum length," etc.

The pages are fairly well supplied with footnotes pointing out sources in which fuller detail of development can be seen, or drawing attention to certain special features of transformations that are worthy of particular attention by the practitioner.

On the whole, the volume is an unusually practicable summary for workers in trigonometry, is singularly free from typographical errors, and is printed in type of such size and variety as to enable the eye to catch readily the particular thing the reader may want. As a ready reference handbook it leaves little to be desired.

G. W. MYERS.

*Beispiel-Sammlung zur Arithmetik und Algebra.* Von Dr. Hermann Schubert, Professor an der Gelehrtenschule des Johanneums in Hamburg. Dritte, durchgesehene Auflage. Leipzig, G. J. Göschen'sche Verlagshandlung, 1905. 136 pp.

THIS is one of the little 80-pfennige handbooks of the well-known Sammlung-Göschen. On 120 pages, 3 inches by 4 inches, it contains 1275 exercises and problems, formal and clothed (eingekleidet), covering the following topics: the transition from calculation to arithmetic, modes of calculating of the first order, of the second order, applications of these modes, quadratics, modes of calculating of the third order, an appendix of problems on higher arithmetic, and selected results.

The appendix contains, in the language of problems: observations on building arithmetic systematically, arithmetic and geometric series, compound interest and annuities, the binomial theorem, Moivre's theorem, and cubic equations.

The problems are arranged in a developmental order, according to the German idea of development. In mathematics the German notion, as shown by their text-book literature, seems to be to begin a topic with a large number of formal and easy exercises, to pass by easy gradations to and through more complicated problems, still of the formal type, and, lastly, to give a list of carefully graded verbal problems having a real content. This accords with the views of some American writers on high school mathematics.

In the view of other writers in our own country and in England, we should get on much better if pupils were not first mechanized by the formal problems. These writers favor fewer of the formal type of problem, and a much larger pro-

portion than is customary of the clothed (eingekleidet) type, the formal exercises to be put at the end of the treatment of topics, to develop mechanical skill, after a larger amount of practice is had in the translation of verbal into formal language. For a considerable time a necessary part of the solution of every problem would be the setting up of the necessary equations. The former mode seeks technique first, then undertakes to infuse thinking into a formal frame-work. The latter seeks to secure the thinking first, and then to develop the technique *as a means of facilitating*, not result-finding, but *thought*. In the reviewer's opinion this collection errs in the over-stressing of the formal side of algebra at the beginning of developments of topics. Of course, teachers may select problems in whatever order they choose. But these lists have evidently been prepared for the special service of teachers whose practice is "through the art to the science."

In this compilation the formal problems outnumber the clothed, or verbal, about in the ratio of 3 to 1. The verbal problems are taken from physical science, from geometry, from elementary mechanics, and from the customary topics of arithmetic. A very large percentage of them are of real modern interest. Any teacher of high school algebra will find this manual a valuable source from which to select exercises to replace many of the dead ones of the standard texts. In this day of correlated algebra and geometry it will be a great help to learn, as these lists of problems show, how many types and varieties of algebraic equations may be based on geometric relations. The chief value of the book for American high school teaching is its adaptability to this service. Many teachers will be glad to find so practical a means as such problems afford of holding the ground made in algebra during the first high school year while the second-year geometry is being taught.

G. W. MYERS.

*An Elementary Treatise on Pure Geometry.* By J. W. RUSSELL, M.A. Oxford, The Clarendon Press, 1905. xii + 366 pp.

THIS is the second edition of a work in which "the author has attempted to bring together all the well-known theorems and examples connected with harmonics, anharmonics, involution, projection (including homology), and reciprocation" (preface to first edition, line 1). In other words, it is con-

cerned with the material of the usual courses on projective geometry, including problems of the first and second degree in the plane. It starts with a metric foundation, Menelaus's and Ceva's theorems, etc., and uses metric geometry freely throughout the book.

Some of the fundamental concepts are introduced in a way that is not likely to impress a beginning student (or any one else) with its logical sharpness. For example, at the bottom of page 9, the notion of "point at infinity" is introduced as follows:

"Take any fixed point $O$ and a fixed line $l$. Then any line $x$ through $O$ cuts $l$ in a point $P$. Now rotate $x$ about $O$ so that $x$ may become more and more nearly parallel to $l$. Then $P$ recedes indefinitely along $l$; and in the limit when $x$ is parallel to $l$, $P$ is said to be the point at infinity upon $l$. Hence *two parallel lines intersect in a point at infinity.*" (Author's italics.)

From here on the "point at infinity" is used quite freely. At the beginning of Chapter III occurs the first mention of imaginary points. It is as follows:

"1. EVERY *line meets a circle in two points, real, coincident, or imaginary.*

"For take any line $l$ cutting a circle in the points $A$ and $B$. Now move $l$ parallel to itself away from the center of the circle. Then $A$ and $B$ approach and ultimately coincide when $l$ touches the circle. But when $l$ moves still further from the center, the points $A$ and $B$ disappear; yet, for the sake of continuity, we say that they still exist, but are *imaginary* (see also XXVII)."

The reference at the end is to Chapter 27, where the aid of algebraic geometry is invoked: "* * * this quadratic equation will have two solutions, real, coincident, or imaginary. Hence we conclude that a line always meets a circle in two points, real, coincident, or imaginary."

It may be true, as many teachers evidently believe, that clear statements on subjects like those to which our quotations apply are too difficult for most elementary students. But on the other hand must be set the distrust inspired in thoughtful students by statements which, taken literally, are untrue, and the flabbiness of mind which comes from arguing with vaguely defined terms.

These criticisms apply only to a limited number of places in Mr. Russell's book and do not in any way impeach its useful-

ness as a source of theorems and examples. The latter are particularly rich in metric cases of projective theorems. The arrangement of material has a number of elegant features as, for example, the way in which theorems on conic sections are derived by projection and reciprocation from the corresponding theorems on circles.

O. VEBLEN.

*Wahrscheinlichkeitsrechnung und Kollektivmasslehre.* By Dr. HEINRICH BRUNS. Leipzig, Teubner, 1906. 310 + 18 pp.

ABOUT one third of this book is devoted to the theory of probability, and two thirds to Kollektivmasslehre. The theory of probability is treated as a theory of frequency, and from this point of view the part on probability is presented in excellent form for application to Kollektivmasslehre. The intimate relation between the two parts of the book stands out so clearly as to make it an important feature, especially because in the work of Fechner Kollektivmasslehre appears much more as an independent subject than as one so closely related to the theory of probability.

While Fechner and Pearson have, to a certain extent, treated Gauss's law of distribution as a "scientific dogma," and have presented generalized probability curves which fit well a large class of data, the conclusion that all distributions conform to one of these curves would have the same kind of logical weakness as the dogma of Gauss. Bearing on this point, Bruns makes a distinct advance by obtaining what seems to be a "suitable" analytic representation for an arbitrary frequency distribution. I use the term "suitable" because it is not difficult to get an analytic representation whose algebraic and numerical complications make it of no value for describing populations such as arise in applications.

Starting with a frequency distribution, the author constructs what he calls a "Summentafel" which gives the number of variates below given values. He uses the term "Summenfunktion" $S(x)$ to represent the relative frequency with which a variate lies below $x$. This function $S(x)$ and its derivative $V(x)$ (Verteilungsfunktion) are the functions for which the author obtains analytic representations. He treats the Summenfunktion as of fundamental importance rather than the distribution function, and in the process of reduction or smoothing which he employs the Summentafel is unchanged, while the frequency distribution may be very much changed.

The analytic representation of the arbitrary Summenfunktion is primarily in terms of two functions, $\phi$ and $R$. The function $\phi(x)$ is defined by

$$\sqrt{\pi}\phi(x) = 2\int_0^x \text{exp. } (-t^2)dt,$$

and the function $R(x)_q$ is given by the recursion formula

$$\text{exp. } (-2xv - v^2) = \sum_{q=0}^{q=\infty} R(x)_q (2v)^q.$$

If the $q$th derivative of $\phi(x)$ is denoted by $\phi(x)_q$ the function $S(y)$ is capable of representation in the form

$$2S(y) - 1 = \sum_{q=0}^{q=\infty} \mathfrak{D}[R(u)_q]\phi(v)_q,$$

where

$$u = h(x - c), \quad v = h(y - c)$$

and the operation $\mathfrak{D}$ is defined, in general, by

$$\mathfrak{D}[T(x)] = \int T(x)\, V(x)dx,$$

where $V(x)$ is the distribution function.

The $c$ and $h$ are parameters so chosen that

$$c = \mathfrak{D}(x), \quad h^2 = \frac{1}{2\mathfrak{D}[x - c]^2}.$$

When thus selected, $c$ is the arithmetic mean and $h$ is the measure of precision.

It is next shown that the above analytic representation gives

$$2S(x) - 1 = \sum_{q=0}^{q=\infty} D(c, h)_q \phi(u)_q.$$

On choosing $c$ and $h$ as just explained,

$$D(c, h)_1 = 0, \quad D(c, h)_2 = 0.$$

Then

$$2S(x) - 1 = \phi(u) + D_3\phi(u)_3 + D_4\phi(u) + \cdots.$$

If the $D_3$, $D_4$, $\cdots$ are zero or so small that they can be neglected, the Verteilungsfunktion gives the simple exponential law. According to this, the author says that populations can

be divided into two large classes: "the first embraces the forms for which the exponential law may be regarded as at least a first approximation; the second, on the contrary, embraces the forms for which this law can not be regarded as even a crude approximation. Thus, the exponential formula, which for a long time was conceived as a sort of law of nature, is put in its proper light. For a large group of distributions arising in Kollektivmasslehre it plays a role similar, say, to the representation of the earth by means of a sphere in measurements of the earth."

In the chapter on the mixture of the arguments, it is shown that this process produces a tendency towards the exponential law. This result is of significance as accounting in part for the large number of distributions which approach this law. In the chapter on the mixture of distributions the essentials of correlation theory are treated.

Of the twenty-four chapters contained in the book, the last five deal with numerical applications. These numerical cases show the systematic methods of carrying the theory into practice, and indicate what parts of the work can be done once for all in many applications.

Dealing with a subject which we should like to see treated rigorously, this book takes a high place in point of mathematical elegance, and it should serve to make much better known this important field of applied mathematics.

H. L. RIETZ.

*Hermann Grassmanns gesammelte mathematische und physikalische Werke.* Herausgegeben von FRIEDRICH ENGEL. Bd. 2: Theil 1, *Die Abhandlungen zur Geometrie und Analysis*, x + 451 pp., 1904; Theil 2, *Die Abhandlungen zur Mechanik und zur mathematischen Physik*, viii + 266 pp., 1902. Leipzig, B. G. Teubner.

THE first volume of Grassmann's works is in two parts, each containing one of the two Ausdehnungslehren; the second volume reprints the miscellaneous papers and Nachlass. It is interesting to note that the earliest paper is a "Programm" on crystals and bears the inscription: Stettin, 1839. This was only five years before the publication of the first Ausdehnungslehre. The last papers are dated 1877 to 1879 and are concerned with various applications of the calculus so intimately associated

with the name of the author. As a matter of fact practically all the intervening matter deals with some phase of mathematics closely allied to one or the other of the Ausdehnungslehren. The author's whole mathematical activity may therefore be said to be confined to this one somewhat narrow and neglected field. The adjective mathematical is, however, a very necessary qualifier of activity; for during the forty years, from 1839 to 1879, Grassmann had been occupied with other researches. In particular, he had published in the early seventies, an exhaustive dictionary to the Rig-Veda, a work more laborious and in its field no less a standard, even to this day, than the two Ausdehnungslehren. To have accomplished such results in two so different fields is a rarity even in a genius.

Of the twenty-four memoirs in the first part of the volume under review, by far the greater number treat the geometric theories of plane cubics and quartics. It is here that one finds the well known methods developed by Grassmann for the construction of plane curves of degree three and four. Of the other contributions perhaps only two deserve individual notice: those entitled "Sur les différents genres de multiplication" and "Stücke aus dem Lehrbuche der Arithmetik." Grassmann's general definition of multiplication is worth quoting here. He says: To multiply extensive magnitudes, first multiply each by each and in the order indicated the units of which the magnitudes are composed; then multiply these products by the products of their respective coefficients and add. He then goes on to state that other products are merely relative. That was in 1859. The standpoint here outlined was that adopted later by Gibbs. In most of Grassmann's work, whether earlier or subsequent, a more specialized position is maintained relative to products. In no other way can one account for the comparatively meager attention given by Grassmann to the dyadics or polyadics (Lückenausdrücke). The selections from a treatise on arithmetic are in reality nothing but a discussion of the theory of numbers taken broadly. It resembles to a great extent the present method, especially Hilbert's method, of laying down axioms or postulates and building up the analysis therefrom. In form the presentation differs considerably from that now adopted. The words postulate and axiom nowhere occur. In their place stands "Erklärung." In the prefatory remarks the author states that certain parts of the treatment are intended for successive grades in the gymnasium. It is difficult to see

how students so immature could profit greatly by the study of such material.

The first memoir on mechanics is again a "Programm" and is intended as an introduction to the subject for students in the gymnasium. Here once more it is difficult to see how the students could have derived much profit from the study. To be sure the presentation is logical and explicit. That seems to be the most serious objection to it. The student would get too much mathematics and logic, too little physics and mechanical intuition from his studies. This would be an excellent argument for having mechanics taught by physicists and engineers rather than by mathematicians, were it not a still better argument for having mathematicians more or less forget their mathematics and really learn mechanics as such before trying to teach it. These two treatises on arithmetic and mechanics are very interesting and highly worth studying. In their way, they are quite distinctive. Grassmann's other work on mechanics is largely a presentation of the subject from the standpoint of the Ausdehnungslehre. The contributions to physics are partly original, partly explanatory. They would not greatly interest anybody now actively engaged in research or in teaching in physical science. This is largely true of all but the highest work done on physics thirty or forty years ago. Mediocrity is short lived in physical science; and it is evident that as a contributor to physics and mechanics Grassmann was not of high rank. This is no reproach: the Ausdehnungslehre and the dictionary to the Rig-Veda are there to silence all specious criticism.

E. B. WILSON.

*Die Grundlagen der Bewegungslehre, von einem modernen Standpunkte aus.* Von DR. G. JAUMANN, Professor der Physik an der Deutschen Technischen Hochschule in Brünn. Mit 124 Abildungen. Leipzig, Johann Ambrosius Barth, 1905. vi + 421 pp. M. 11; M. 12 gebunden.

IN the author's view the transformation which Fresnel wrought in the theory of light, and Faraday in electromagnetic phenomena, furnishes a prototype for all domains of theoretical physics, mechanics not excepted. The latter, although preserving its old form unaltered, proceeds now in part from points of view which lay quite beyond the reach of its founder and, in this respect, it appears to be nearing a decisive change. This rather remote aim gave direction to the treatment of this book.

It is held that the classical method of deducing a science from the minimum number of experiments and a few fundamental propositions is no longer best; if, indeed, it is admissible at all nowadays. That method furnishes deductive certainty and breadth of development, but it closed the way to progress. The somewhat rigid, dogmatic form of present-day mechanics is due to the manner of its founding. Newton based it upon the very special domain of the motion of discrete rigid bodies in the air. Though this procedure is historically explicable, it can hardly endure indefinitely.

The form of the treatment chosen is that of a systematic textbook of the foundations of mechanics. The pedagogical purpose, of course only a secondary one, has been pursued consistently and seriously. In every appropriate place, the attempt is to make the treatment as easily comprehensible as possible. But where rigor and clearness in main questions called for it, some higher demands have been made upon the reader.

The same point of view is taken regarding the analytic means employed. Advance in theoretical physics depends materially upon the development of suitable mathematical notions. The question of mechanics is always that of disclosing a law to bind observations together. An empirical law for a special series of observations and a general physical law proceed from essentially similar inductions. To discover the form of an empirical law requires choice from amongst appropriate mathematical forms. Without this only naive theories can arise and many geometric relationships must be regarded as erroneous for the fundamental facts of physics.

Reasoning like the foregoing brings the author to the conclusion that it is impossible to desist from employing vector analysis. To avoid narrowing his circle of readers, the author distributes through the book in appropriate places what, in its totality, makes up a brief exposition of the whole theory of vector analysis. The citations for fuller elucidation of the theory are to Gibb's Vector Analysis, edited by E. B. Wilson, New York, 1904, whence the greater part of the notation is taken.

A brief and suggestive introduction sketches lucidly the psychological origin of the mental bias of man which leads him to over-emphasis of the relative importance of mechanical facts and laws as compared with other domains of natural law. As an illustration, the circumstance is cited that on account of the

one-sided development of our senses, we unconsciously ascribe greater significance to the change involved in raising a mass of water 424 meters high, or throwing it with an initial velocity of 91 meters per second, than we do to the change involved in heating the mass of water enough to raise its temperature by 1° centigrade, although these effects have been proved to be objectively equivalent.

The compact language and modes of thought of vector analysis make possible a very condensed treatment of a very extended field of phenomena. The topics that are covered with considerable fulness of detail are the following: motion of rigid bodies in the air, dynamic reaction, rigid media, acoustics (embracing pure acoustics and vibroscopy), motion of deformable media, ideal fluids, elastic and viscous media, and the doctrine of force, including forces in space, and forces on surfaces.

For those who have not accustomed themselves to the form of thought of vector analysis, the book will offer some difficulties, despite the author's attempt to simplify matters, but the elegance of the new mode of scientific thought will repay the effort needed to overcome the difficulties, which are not great. The book is quite as interesting and informing in its interpretations of the various results of the operations of vector analysis, as for its value as a higher presentation of the theory of motion.

The author has made a worthy and a more than fairly successful attempt to do a laudable scientific service. He seeks to bring into organic union the most powerful mathematical language and form of thought yet devised, and a broad field of scientific ideas demanding precisely this sort of language for its adequate expression. The undertaking is too well carried out to merit anything but commendation. To complain of a few typographical errors would only augment the volume of hysteria for criticism already too rampant in certain quarters among us.

G. W. MYERS.

*Text-Book on the Strength of Materials.* By S. E. SLOCUM, B.E., Ph.D., Professor of Applied Mathematics in the University of Cincinnati, and E. L. HANCOCK, M. S., Assistant Professor of Applied Mechanics in Purdue University. Ginn and Co., 1906. xii + 314 pp.

THIS is one of the series of mathematical texts that are being issued under the editorial supervision of Professor Percey

F. Smith, of the Sheffield Scientific School. Both as to content and form this new addition is up to the high standard of excellence for which the series has already earned a reputation.

The authors have aimed to make a text-book on the strength of materials, which should represent the best theory and practice and at the same time be elementary enough for junior classes in technical and engineering schools. To this end they have devoted the first part of the book to theoretical discussions for class use, while the second part is a description of the physical properties of materials, with methods of testing, etc., the latter being intended for use as a sort of laboratory manual. To arouse interest in the subject, practical applications are introduced in Part I, for computation; and in Part II, for observation.

The following phases of the subject are stressed: the definition of the moment of inertia as the shape factor; the graphical method of calculating moments of inertia and centers of gravity; the application of the principle of least work; comparison of column formulas, with graphical illustrations of their relation; precise formulas for torsion of shafts of various cross-sections; simple and correct methods for the strength of hooks, links, springs, etc.; simple formulas for the strength of flat plates; elementary discussion of arches, of retaining walls and foundations; and a separate chapter in Part II on the modern use of reënforced concrete.

The question of most serious import to the writer is whether, on the whole, it is better to divide the student's time, which is now usually given to strength of materials alone, between that science and what is commonly taught in a course by itself under the heading Materials of Construction. Briefly, the book seems trying to do too much. Those who, like the writer, have taught strength of materials to junior engineering students have not found the allotted time any too great for that subject alone. As a student, the writer well recalls that the same truth holds for materials of construction. Even with a full course on the latter subject, but little more than a mere beginning can be made. So brief a treatment as is necessitated by making it a subordinate part of a course in strength of materials would hardly furnish a smattering. Both of these subjects are of fundamental importance to prospective engineers, and neither ought to be curtailed to the extent of giving them both in a single course, unless that

course cover an entire year. The practicability of this extension of time is against it. Furthermore, the book does not go into the subject far enough to justify this expenditure of time on the combined course. In the reviewer's opinion, the book has fallen into the inevitable error of too great condensation, viz., superficiality.

The correlation of these two subjects treated is, of course, important. It is also readily admitted that differentiation of subject matters is a necessary part of any true correlation. But the treatment of two subjects separately, one after the other, is not correlating them at all. It is only getting ready for a sound correlation. A real correlation of the subjects would mean furnishing the observational and experimental backing to the theory antecedently, and in close connection with it; and not subsequently and in no particularly close relation to the theory. From the viewpoint of teachability the writer is disposed to question the separation of the subjects, the order of them after they are separated, and the limited scope of both parts of the work.

The work that is given is exquisitely well done. The demonstrations are simple and graphic. It can hardly be maintained that there is either too much or too little illustrating and exemplifying. In these regards the book is unusually well-balanced. Mathematics is used, and freely used, when it helps. So far as the book goes it is mathematical enough to justify the claim to being truly, not naïvely, scientific. Every teacher would find great assistance from it, and many schools will find it well adapted to their purposes and programs.

As a mechanical piece of book-making the text is nothing short of admirable. The cuts — many of them very difficult — the typography, and the arrangement of matter on the page are up to the high standard of the contents of the book.

G. W. Myers.

## NOTES.

THE July number (volume 8, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "General theory of approximation by functions involving a given number of arbitrary parameters," by J. W. YOUNG; "On derivatives over assemblages," by E. R. HEDRICK; "Geometrie proiettive di congruenza e geometrie proiettive finite," by B. LEVI; "Collineations in a finite projective geometry," by O. VEBLEN; "Geometry in which the sum of the angles of every triangle is two right angles," by R. L. MOORE; "Non-desarguesian and non-pascalian geometries," by O. VEBLEN and J. H. MACLAGAN-WEDDERBURN; "Modular theory of group matrices," by L. E. DICKSON; "Existence proof for a field of extremals tangent to a given curve," by O. BOLZA; "A new form of the simplest problem of the calculus of variations," by G. A. BLISS; "On certain isothermic surfaces," by A. E. YOUNG.

THE University of Illinois has been added to the list of institutions contributing to the support of the *Transactions*.

THE concluding (July) number of volume 8 of the *Annals of Mathematics* contains: "Multiply perfect numbers of four different primes," by R. D. CARMICHAEL; "On a system of parastroids," by R. P. STEPHENS; "A peculiar example in minima of surfaces," by E. R. HEDRICK: "On maximum and minimum values of the modulus of a polynomial," by D. N. LEHMER; "On the minimum surface of revolution in the case of one variable end point," by Miss M. E. SINCLAIR; "On the polynomial convergents of a power series," by M. B. PORTER.

THE annual list of American doctorates published in *Science* presents for the academic year 1906–1907 327 names, of which 168 are credited to the sciences. The following 13 successful candidates offered mathematics as major subject (the titles of the theses are appended): Miss F. A. ALLEN, Wisconsin, "On the determination of cyclic involutions of order three"; C. S. ATCHISON, Johns Hopkins, "Curves with a directrix"; G. D. BIRKHOFF, Chicago, "Asymptotic properties of certain ordinary differential equations with applications to boundary value and expansion problems"; W. C. BRENKE, Harvard, "A con-

tribution to the theory of trigonometric and zonal harmonic series"; H. H. CONOVER, Yale, "On certain problems in the calculus of variations"; A. E. LANDRY, Johns Hopkins, "A geometric interpretation of binary syzygies"; R. MORRIS, Cornell, "On the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$"; F. W. REED, Virginia, "Singular points in the approximate development of the perturbation function"; A. RANUM, Chicago, "A new kind of congruence groups"; F. R. SHARPE, Cornell, "The general circulation of the atmosphere"; W. B. STONE, Virginia, "The groups of two, three, and four parameters in space and their differential invariants"; A. L. UNDERHILL, Chicago, "Invariants under point transformations in the calculus of variations"; B. M. WALKER, Chicago, "On the resolution of higher singularities of algebraic curves into ordinary double points."

The number of American doctorates in mathematics for each year of the last decade is 11, 13, 11, 18, 8, 7, 14, 21, 11, 13.

THE last annual list of members of the London mathematical society contains 280 names, including 27 Americans and 13 honorary members. The society was founded in 1865, as the outgrowth of the students' mathematical clubs, to a large extent through the efforts of Professor A. DeMorgan, who was made its first president. The last of the original charter members was the late Dr. E. J. Routh. Steps were at once taken to publish papers read before the society, the first volume of the *Proceedings* appearing in 1870, and a volume has appeared every year since. An index of authors of papers published in the first thirty volumes was issued in 1900; in 1902 the size of the page was increased, and larger type employed. In this new series the reports of the meetings appear at the beginning of the volume. The list of papers read at each monthly meeting has regularly appeared in the BULLETIN. In honor of its first president, the society in 1884 founded the DeMorgan medal, which is awarded every three years for exceptional merit. The list of medallists is Professors Cayley, Sylvester, Lord Rayleigh, Klein, S. Roberts, W. Burnside, Greenhill, Baker. The next award will be made in 1908. The officers for the present year are: president, Professor W. BURNSIDE; secretaries, Professor A. E. H. LOVE and Mr. J. H. GRACE; two vice-presidents, treasurer and ten other members of the council. The next election will he held November 14, 1907.

THE *Revista trimestral de Matemáticas*, founded six years ago by Professor J. RUIS Y CASAS of Zaragoza, was discontinued with the completion of number 21, which appeared last spring. In its place a new periodical has been founded, with the title, *Anales de la Facultad de Ciencias de Zaragoza*, which will contain memoirs from the sciences of mathematics, physics, chemistry, astronomy, meteorology and the natural sciences.

THE academy of sciences of Copenhagen announces the following prize problem, for the solution of which it will award the gold medal of the society: "To complete, by new results, the theory of the plane or of an algebraic surface, the points of which correspond reciprocally." Competing memoirs must be written in Danish, Swedish, English, German, French or Latin, and sent to the secretary, Professor H. G. Zeuthen, before October 31, 1908.

THE various German universities offer the following courses in mathematics during the winter semester of 1907–08.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic geometry, four hours; Applications of elliptic functions, four hours; Examples in conformal mapping, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of curves and surfaces, four hours; Linear differential equations and automorphic functions, four hours; Seminar, two hours. — By Professor G. HETTNER: Infinite series, products and continued products, two hours. — By Professor J. KNOBLAUCH: Differential calculus, four hours; Theory of elliptic functions, four hours. — By Professor E. LANDAU: Integral equations, four hours. — By Dr. I. SCHUR: Theory of algebraic equations, four hours; Linear substitutions, two hours. — By Professor R. LEHMANN-FILHÈS: Integral calculus, four hours. — By Dr. J. ASCHKINASS: Elements of higher mathematics for students of science, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Introduction to the theory of functions, four hours; Introduction to quaternions, one hour; Seminar, two hours. — By Professor F. LONDON: Advanced calculus, four hours; with exercises, one hour; Synthetic geometry, two hours; Exercises in descriptive geometry, two hours; Seminar, two hours. — By Professor G.

KOWALEWSKI: Analytic geometry of the plane and of space, four hours; with exercises, one hour; Introduction to the theory of groups of transformations, two hours.— By Dr. E. SCHMIDT: Partial differential equations, four hours.

UNIVERSITY OF GÖTTINGEN.— By Professor F. KLEIN: Elementary mathematics from a higher standpoint, four hours; Seminar, two hours.— By Professor D. HILBERT: Theory of partial differential equations, four hours; Integral equations, two hours; Seminar, two hours.— By Professor H. MINKOWSKI: Theory of functions, four hours; Selected chapters of the theory of numbers, two hours; Seminar, two hours.— By Professor C. RUNGE: Graphical methods, four hours; with exercises, two hours; Seminar, two hours.— By ——: The basis of insurance, three hours; Seminar.— By Professor E. ZERMELO: The mathematical basis of logic, two hours. — By Dr. G. HERGLOTZ: Algebra, four hours; Exercises in differential equations, two hours.— By Dr. C. CARATHÉODORY: Advanced calculus, four hours; with exercises, two hours.— By Dr. P. KOEBE: Quadric surfaces, two hours.— By Professor W. VOIGT: Mechanics, four hours.— By Dr. M. ABRAHAM: Theory of elasticity, two hours.

UNIVERSITY OF HALLE. — By Professor G. CANTOR: Theory of elliptic functions, four hours; Selected chapters of analytic mechanics, two hours; Seminar, two hours — By Professor A. WANGERIN: Theory of potential and spherical harmonics, four hours; Integral calculus with exercises, four hours; Seminar, two hours. — By Professor A. GUTZMER: Calculus of variations, four hours; Analytic geometry of space, four hours; Axonometry and perspective, two hours. — By Professor V. EBERHARD: Numerical equations and iterated functions, two hours; Theory of numbers, II, two hours; Colloquium, two hours. — By Dr. F. BERNSTEIN: Theory and application of definite integrals, four hours; Mathematics of insurance, one hour.

UNIVERSITY OF JENA. — By Professor F. HAUSSNER: Integral calculus with exercises, five hours; Algebra, four hours; Analytic geometry of space, four hours; Seminar, two hours; Proseminar, two hours. — By Professor J. THOMAE: Differential equations, four hours; Seminar, two hours. — By Professor G. FREGE: Analytic mechanics, four hours; Symbolic methods, one hour. — By Professor R. RAU: Graphical methods in statics and mechanics, three hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Analytic mechanics, four hours. — By Professor A. MAYER: Calculus of variations, four hours. — By Professor O. HÖLDER: Elliptic functions, four hours; Seminar, two hours. — By Professor K. ROHN: Analytic geometry of space, four hours; Descriptive geometry, two hours; with exercises, two hours. — By Professor F. HAUSDORFF: Calculus, four hours; with exercises, two hours. — By Professor H. LIEBMANN: Theory and application of determinants, two hours; Non-euclidean geometry, two hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Differential calculus, five hours; Theory of abelian functions, four hours; Seminar, one and a half hours. — By Professor A. VOSS: Plane analytic geometry, four hours; Analytic mechanics, I, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Theory of infinite series and of similar limiting processes, four hours; Elements of the theory of functions, five hours. — By Professor K. DOEHLEMANN: Descriptive geometry, I, five hours; with exercises, three hours; Synthetic geometry, with exercises, five hours; Lines in art, one hour. — By Professor H. BRUNN; Theory of aggregates, four hours. — By Dr. K. HARTOGS: Integral calculus, six hours. — By Dr. O. PERRON: Elementary geometry and trigonometry, three hours.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: New methods in the analytic geometry of space, four hours; Mathematical theory of elasticity of rigid bodies, two hours; Seminar, two hours. — By Professor H. WEBER: Calculus, four hours; Differential equations of mathematical physics, two hours; Seminar, hours. — By Professor M. SIMON; Fundamental conceptions of mathematics and mechanics, four hours. — By Professor J. WELISTEIN: Selected chapters of the theory of functions, four hours; Determinants and matrices, two hours; Seminar, two hours. — By Professor H. E. TIMERDING: Plane analytic geometry, with exercises, four hours; Graphical statics with exercises, two hours; Vector analysis, two hours; Seminar, two hours. — By Dr. S. EPSTEIN: Introduction to higher mathematics for students of natural science, four hours; Seminar, two hours.

AT the University of Göttingen Professor L. PRANDTL has been promoted to a full professorship of mathematics; Professor

M. BRENDEL has been appointed professor of mathematics at the commercial academy of Frankfurt am Main; Dr. O. TOEPLITZ has been appointed docent in mathematics.

PROFESSOR J. DRACH, of the University of Poitiers, has been transferred to a professorship of analysis at the same institution.

DR. M. GROSSMANN, of the University of Basel, has been appointed professor of descriptive geometry at the technical of Zürich, as successor to Professor W. Fiedler, who has retired from active service.

DR. F. HAUSDORFF has been elected an associate member of the academy of sciences at Leipzig.

PROFESSOR J. HORN, of the mining academy at Clausthal, has been appointed professor of mathematics at the technical school of Darmstadt.

DR. L. SCHRUTKA, has been appointed docent in mathematics at the University of Vienna.

PROFESSOR G. HESSENBERG, of the technical school at Charlottenburg, has been appointed professer of mathematics at the agricultural institute at Bonn-Poppelsdorf.

PROFESSOR A. KORN, of the University of Munich, has been elected a member of the academy of sciences at Halle.

DR. M. KUTTA has been promoted to an associate professorship of mathemaics at the technical school of Munich.

PROFESSOR R. D. v. STERNECK, of the University of Czernowitz, has been appointed professor of mathematics at the University of Graz.

DR. R. WEBER, of the University of Heidelberg, has been appointed associate professor of mathematics at the University of Rostock.

DR. G. HERGLOTZ, of the University of Göttingen, has been appointed associate professor of mathematics at the University of Freiburg, Switzerland.

PROFESSOR H. v. MANGOLDT, of the technical school at Danzig, has been decorated with the order of the red eagle of the third class. Professor M. DISTELI, of the technical school at Dresden, has received the title of knight of the cross, of the first class.

THE following promotions to full professorships of mathematics have been made in the French provincial universities: Professor J. CLAIRIN, at the University of Lille; Professor R. LE VAVASSEUR, at the University of Lyons; Professor A. PARAF, at the University of Toulouse.

MR. J. MERCER has been appointed assistant lecturer in mathematics at the University of Liverpool.

AT the University of Illinois the following changes have been made in the mathematical staff: Professor G. A. MILLER has been promoted to a full professorship; Professor E. J. WILCZYNSKI, formerly at the University of California, has been appointed associate professor; Dr. C. H. SISAM was advanced from the grade of instructor to associate; Mr. B. M. MATHEWS has been appointed assistant. As at present constituted the department consists of three professors, one associate professor, three assistant professors, one associate, five instructors, and three assistants.

THE following changes are announced at the University of Missouri: Dr. W. D. A. WESTFALL has been appointed assistant professor of mathematics and granted leave of absence for one year to study in Europe; Dr. OTTO DUNKEL has been appointed instructor; Dr. L. INGOLD returns from his leave of absence; Mr. L. L. SILVERMAN has been appointed acting instructor; Miss A. PAYNE and Miss A. M. LIEPSNER have been appointed assistants; Miss M. S. WALKER has been appointed instructor and granted leave of absence to continue her studies.

PROFESSOR T. F. HOLGATE, of Northwestern University, has been granted leave of absence for the present academic year, and will spend most of the time at Cambridge, England. Professor D. R. CURTISS has been promoted to a full professorship of mathematics.

DR. A. B. PIERCE, of the University of Michigan, has been promoted to an assistant professorship of mathematics.

PROFESSOR O. P. AKERS, of Allegheny College, has been made full professor of mathematics.

DR. J. A. EIESLAND has been promoted to the chair of mathematics at the United States Naval Academy, Annapolis.

DR. W. C. BRENKE has been appointed adjunct professor of mathematics at the University of Nebraska.

PROFESSOR G. D. GABLE, of Parsons College (Iowa), has been appointed professor of mathematics at the University of Wooster, Ohio.

DR. W. H. BUSSEY, of Columbia University, has been appointed assistant professor of mathematics at the University of Minnesota.

PROFESSOR A. L. RHOTON, of the Southwestern University of Tennessee, has been appointed professor of mathematics at Georgetown College, Kentucky.

PROFESSOR R. C. ARCHIBALD, of the Mount Allison Ladies' College, has been appointed professor of mathematics at Acadia University, Wolfsville, N. S.

THE following appointments have been made at the University of Pennsylvania: Dr. F. H. SAFFORD has been promoted to an assistant professorship of mathematics; Messrs. M. J. BABB and L. O'SHAUGHNESSY have been named instructors in mathematics.

DR. R. P. STEPHENS, of Wesleyan University, has been appointed adjunct professor of mathematics at the University of Georgia. Mr. H. B. CAMP has been appointed instructor in mathematics at Wesleyan.

AT Syracuse University, Mr. D. PRATT has been appointed assistant professor of mathematics, and Mr. H. F. HART has been appointed instructor in mathematics.

DR. E. SWIFT has been appointed instructor in mathematics at Princeton University.

MR. R. L. BÖRGER, of the University of Florida, has been appointed instructor in mathematics at the University of Illinois.

MR. G. I. GAVETT, of Cornell University, has been appointed instructor in mathematics at the University of Washington.

MR. A. C. KRATHWOHL has been appointed tutor in mathematics at Barnard College, Columbia University.

MR. C. A. TOUSSAINT, of Columbia University, has been appointed tutor in mathematics at the College of the City of New York.

MR. W. V. LOVITT has been appointed instructor in mathematics at the University of Washington, Seattle, Washington.

MR. W. T. AUDE, of Colgate University, has been appointed instructor in mathematics at the Carnegie Institute in Pittsburgh.

MR. C. H. FORSYTH has been made assistant in mathematics at the University of Illinois.

MISS M. E. SINCLAIR, of the University of Nebraska, has been appointed instructor in mathematics at Oberlin College.

PROFESSOR E. C. COLPITTS, of the Georgia School of Technology, has been appointed professor of mathematics at Emporia College, Kansas.

DR. OREN ROOT, for 27 years professor of mathematics at Hamilton College, died August 26, at the age of 69 years.

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## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

BROGGI (U.). Traité des assurances sur la vie avec développements sur le calcul des probabilités. Paris, Hermann, 1907. 8vo. 11+309 pp.

DINI (U.). Lezioni di analisi infinitesimale. Vol. I. Calcolo differenziale. Pisa, Nistri, 1907. 8vo. 720 pp. L. 22.00

DOLGE (P.). Ueber Bernoullische Zahlen und Funktionen, welche zu einer Fundamentaldiskriminante gehören, und deren Anwendung auf die Summation unendlicher Reihen. (Progr.) Dresden, 1907. 8vo. 44 pp.

EGERER (H.). Ueber die Kurve der Ecken der Vierseite, die von den gemeinsamen Tangenten eines festen Kegelschnitts und der Kegelschnitte eines Büschels gebildet werden. (Diss.) Erlangen, 1906. 8vo. 53 pp.

FACK (M.). Zur didaktischen Darstellung von Stoffen aus der niedern und höheren Mathematik. (Progr.) Weimar, 1907. 8vo. 53 pp.

FAZZARI (G.). Breve storia della matematica dai tempi antichi al medio evo. Palermo, Sandron, 1907. 16mo. 267 pp. L. 4.00

FOETHKE (E.). Anwendung des erweiterten Euklidischen Algorithmus auf Resultantenbildungen. (Diss.) Königsberg, 1907. 8vo. 75 pp.

GILLESPIE (D. C.). Anwendungen des Unabhängigkeitssatzes auf die Lösung der Differentialgleichungen der Variationsrechnung. (Diss.) Göttingen, 1906. 8vo. 75 pp.

GRABER. Zur Zylinder- und Kugelberechnung. (Progr.) Kottbus, 1907. 8vo. 39 pp.

GUTSCHE (O.). Geometrische Untersuchungen über inverse Kurven von Kegelschnitten. (Progr.) Breslau, 1907. 8vo. 44 pp.

KRUSCHE (A.). Ueber Kurven und Flächen, welche sich aus geradlinigen Flächen 2 ten Grades durch gemeinsame Lote zwischen den Erzeugenden ableiten lassen. (Diss.) Breslau, 1905. 8vo. 58 pp.

LORIA (G.). Vorlesungen über darstellende Geometrie. Autorisierte, nach dem italienischen Manuskript bearbeitete deutsche Ausgabe von F. Schütte. Teil I: Die Darstellungsmethoden. Leipzig, Teubner, 1907. 8vo. 11 + 219 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern, Vol. XXV.) M. 6.80

MEISSNER (W.). Zur Theorie des Strahlungsdruckes. (Diss.) Berlin, 1907. 8vo. 75 pp.

OPPLIGER (A.). Historisch-kritische Untersuchung über die Theorie der Kugelfunktionen. (Diss.) Bern, 1906. 8vo. 62 pp.

PEREZ DE MUNOZ (F.). Introducción al estudio del cálculo de Cuaterniones y otras algebras especiales. Madrid, Garcia, 1907. 8vo.

PETERS (J. M.). Die Summen der ganzen positiven Potenzen der natürlichen Zahlen und ihre Darstellung als Funktionen der Summe dieser Zahlen. (Progr.) Cöln, 1907. 4to. 21 pp.

PYRKOSCH. Das Büschel von Kurven 4ter Ordnung mit 3 festen Doppelpunkten und 4 festen einfachen Punkten. (Progr.) Breslau, 1907. 8vo. 16 pp.

RICHERT (P.). Die ganzen rationalen Funktionen der ersten drei Grade und ihre Kurven. Exponentialreihen höherer Grade. (Progr.) Berlin, Weidmann, 1907. 8vo. 77 pp. M. 1.00

WHITEHEAD (A. N.). The axioms of descriptive geometry. Cambridge, University Press, 1907. 8vo. 82 pp. 2s. 6d.

WIRTH (H.). Beiträge zur Theorie der Abbildungen durch reciproke radii vectores. (Progr.) Wolgast, 1907. 8vo. 32 pp.

ZEHME. Eigenschaften des Krümmungsschwerpunktes ebener Kurven. Teil II. (Progr.) Arnstadt, 1907. 4to. 24 pp.

## II. ELEMENTARY MATHEMATICS.

ARNOUX (V.). Deux mille cinq cents problèmes d'arithmétique et de géométrie pratique. (Nombres décimaux, fractions, règles de trois simple et composée, intérêts simple et composés, rentes sur l'état, etc.). Problèmes d'examen. 10e édition, revue et augmentée. Paris, Larousse, 1907. 12mo. 288 pp. F. 2.00

BAKER (W. M.). A key to algebraic geometry. London, Bell, 1907. 8vo. 7s. 6d.

BARNARD (S.) and CHILD (J. M.). A new geometry for middle forms, being parts 1, 2 and 3 of "A New Geometry" and equivalent to Euclid, Books 1–4, together with additional matter. London, Macmillan, 1907. 12mo. 438 pp. 3s. 6d.

BEARD (W. S.). Easy exercises in algebra for beginners. London, Methuen, 1907. 12mo. 118 pp. 1s. 6d.

BOURLET (C.). Eléments d'algèbre, rédigés conformément aux programmes du 31 mai 1902 et aux programmes de l'enseignement primaire supérieur. 2e édition. Paris, Hachette, 1907. 16mo. 388 pp. F. 3.00

BROOKSMITH (E. J.). Mathematical papers. For admission into the Royal military academy and the Royal military college, for the years 1897–1906. London, Macmillan, 1907. 12mo. 6s.

CHILD (J. M.). See BARNARD (S.).

DEAKIN (R.). Key to the new matriculation algebra. London, Clive, 1907. 12 mo. 342 pp. 5s. 6d.

——. New geometry papers. London, Clive, 1907. 12mo. 1s.

ERNEWELL (E. R.). Regeln der Mathematik. I. Lehrgang. Die Geometrie der Ebene (Planimetrie) und die Grundlehren der Stereometrie (Raumlehre) für den Schul- und Selbstunterricht. Nach völlig neuer, streng wissenschaftlicher Methode für jedermann verständlich dargestellt. Berlin, Selbstverlag, 1907. 8vo. 4 + 100 pp. M. 10.00

GREVY (A.). Géométrie dans l'espace à l'usage des élèves des classes de première C et D (Programme du 27 juillet, 1905). 2e édition. Paris, Vuibert, 1907. 8vo. 214 pp. F. 2.00

JACKSON (W. H.). Elementary solid geometry, including the mensuration of the simple solids. New York, Longmans, 1907. 12mo. 12+159 pp. $0.65

KELLER (S. M.). Mathematics for engineering students: algebra and trigonometry. New York, Van Nostrand, 1907. 8vo. 4 + 250 pp. (Carnegie Technical Schools text-books). Cloth. $1.50

KÖHLER (A.). See LIEBER (H.).

KUTNEWSKY (M.). See MÜLLER (H.).

LAZZERI (G.) e PESCHI (G.). Complementi d'algebra per l'ammissione alla r. accademia navale. 2e edizione. Livorno, Unione poligrafica livornese, 1907. 8vo. 120 pp. L. 1.50

LEBON (E.). Géométrie cotée et géométrie descriptive (conforme aux programmes du 27 juillet 1905 pour l'enseignement secondaire). 2e édition. Paris, Delalain, 1907. 8vo. 198 pp.

——. Sur la construction d'une table de caractéristiques relatives à la base 30030 des facteurs premiers d'un nombre inférieur à 901800900. Paris, Imprimerie nationale, 1907. 8vo. 7 pp.

LIEBER (H.) und KÖHLER (A.). Arithmetische Aufgaben. 4te Auflage. Berlin, Simion, 1907. 8vo. 8 + 232 pp. M. 2.70

MARKS (C. I.). Mathematical questions and solutions. New Series, Vol. II. London, Hodgson, 1907. 8vo. 6s. 6d.

MICHALITSCHKE (A.). Die mathematische Geographie und die Himmelskunde im elementaren Unterrichte an der Volks- und an der Mittelschule. Ein Beitrag zur Methodik des Gegenstandes. Wien, Deuticke, 1907. 8vo. 3 + 24 pp. M. 0.80

MÜLLER (H.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe B, für reale Anstalten und Reformschulen. Teil II: 2te Auflage. Leipzig, Teubner, 1907. 8vo. 11+304 pp. Cloth. M. 3.00

PESCHI (G.). See LAZZERI (G.).

RICHTER (O.). Dreistellige logarithmische und trigonometrische Tafeln. Leipzig, Teubner, 1907. 8vo. 10 pp. M. 0.20

SCHULZ (E.). Leitfaden der Planimetrie für gewerbliche Lehranstalten. Teil II. 4te Auflage. Essen, Baedeker, 1907. 8vo. 4+98 pp. Boards M. 1.00

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